Print Your Name Here: ______________________________

Show all work in the space provided. *Indicate clearly* if you continue on the back. Write your **name** at the **top** of the *scratch* sheet *if you will hand it in to be graded.* **No** books, notes, smart phones, cell phones, communication devices, internet devices, or electronic devices are allowed except for a scientific calculator—which is not needed. The maximum total score is 100.

Part I: Short Questions. Answer **8** of the 12 short questions: 6 points each. **Circle** the **numbers** of the 8 questions that you want counted—*no more than 8*! Detailed explanations are not required, but they may help with partial credit and are *risk-free!* Maximum score: 48 points.

1. Find the *largest value* for the number $\delta > 0$ that is *small enough* so that $|a-1| < \delta$ and $|c-1| < \delta$ implies $|a-c| < \frac{1}{20}$.

2. Give an example of three vectors $\mathbf{a},\ \mathbf{b},\ \mathbf{c}$ in 3 dimensional space such that $(\mathbf{a}\times\mathbf{b})\times\mathbf{c} \neq \mathbf{a}\times(\mathbf{b}\times\mathbf{c})$. Show the values for the two triple cross products.

3. The sequence x_n follows this repeating pattern: $\mathbf{0}, \mathbf{1}, \frac{1}{2}, \mathbf{0}, \frac{1}{3}, \frac{2}{3}, \mathbf{1}, \frac{3}{4}, \frac{1}{2}, \frac{1}{4}, \mathbf{0}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \mathbf{1}, \ldots$. True or False: x_n is a Cauchy sequence.

4. Let $x_n = (-1)^n \frac{n}{n+1}$ for all $n \in \mathbb{N}$ and let T_N be the Nth *tail* of this sequence. Find $s_N = \sup(T_N)$ and $i_N = \inf(T_N)$.

5. True or Give a Counterexample: If x_n and y_n are both monotone increasing, then $x_n y_n$ is monotone.

6. Give an example of two sequences $x_n \to \infty$ and $y_n \to \infty$ such that $\dfrac{x_n}{y_n} \to \pi$.

7. Find $\displaystyle\bigcap_{n=1}^{\infty} \left(1 - \frac{1}{n}, 1\right)$.

8. Give an example of a decreasing nest of infinitely long intervals I_n with empty intersection.

9. Let $E \subseteq \mathbb{R}$ be any *unbounded* set. Find an open cover $\mathcal{O} = \{O_n \mid n \in \mathbb{N}\}$ of E that has no finite subcover.

10. Give an example of sequences x_n and y_n with $\limsup(x_n + y_n) < \limsup x_n + \limsup y_n$.

11. True or Give a Counter-example: If x_n is not a bounded sequence, Then $|x_n| \to \infty$.

12. Let x_n be a sequence of real numbers with the property that the sequence has *no least term.* True or False: The sequence x_n has a *strictly decreasing* subsequence: $x_{n_k} \downarrow$ as k increases.

Part II: Proofs. *Prove carefully* **2** of the following 3 theorems for 26 points each. **Circle** the *letters* of the 2 proofs to be counted in the list below—*no more than 2!* You may write the proofs below, on the back, or on scratch paper. Maximum total credit: 52 points.

A. Prove: If $s_n \le t_n \le u_n$ for all $n \in \mathbb{N}$ and if both $s_n \to L$ and $u_n \to L$ then $t_n \to L$ as $n \to \infty$ as well. (This is sometimes called the *squeeze theorem for sequences*.)

B. Suppose A and B are subsets of $\mathbb{R}$, both *nonempty*, with the special property that $a \le b$ for all $a \in A$ and for all $b \in B$. Prove: $\sup(A) \le \inf(B)$. (Hint: How does each $b \in B$ relate to the set A?)

C. Suppose that the sequence x_n has *no convergent subsequences*. Let $M > 0$. Prove that there exists $N \in \mathbb{N}$ such that $n \ge N \implies x_n \notin [-M, M]$. Explain why this implies $|x_n| \to \infty$ as $n \to \infty$.

Solutions and Class Statistics

1. $\delta = \dfrac{1}{40}$, since by the triangle inequality we need $2\delta = \dfrac{1}{20}$.

2. For example, $(\mathbf{i} \times \mathbf{j}) \times \mathbf{j} = -\mathbf{i} \neq \mathbf{0} = \mathbf{i} \times (\mathbf{j} \times \mathbf{j})$. Always try first for an easy example with which to calculate!

3. False. No matter how large $N \in \mathbb{N}$ is there will be n, $m \geq N$ such that $|x_n - x_m| = |1 - 0| = 1$.

4. $s_N = 1$ and $i_N = -1$ for all $N \in \mathbb{N}$. We remark that $\limsup x_n = 1$ and $\liminf x_n = -1$.

5. Counterexample: Let $x_n = n - 4 = y_n$. There are many such examples: the issue is with sequences that are not identically positive.

6. For example, let $x_n = n\pi$ and $y_n = n$.

7. $\displaystyle\bigcap_{n=1}^{\infty} \left(1 - \frac{1}{n}, 1\right) = \emptyset$.

8. For example, let $I_n = [n, \infty)$ for each $n \in \mathbb{N}$.

9. For example, let $O_n = (-n, n)$ for each $n \in \mathbb{N}$.

10. For example, let $x_n = (-1)^n$ and $y_n = (-1)^{n+1}$.

11. Counter-example: consider the sequence x_n for which $x_n = 0$ if n is *odd* but $x_n = n$ if n is *even*.

12. True: See the second part of the proof of the existence of monotone subsequences, in the homework.

Remarks about the proofs

Proofs are graded for logical coherence. If you have questions about the grading of the proofs on this test, or if you are having difficulty writing satisfactory proofs, *please bring me your test and also the graded homework from which the questions in Part II came.* This will help us to see how you use the corrections to your homework in order to learn to write better proofs. Also *please bring your notebook showing how we presented the same proof in class* after the homework was graded. It is important to learn from both sources.

A: Most students wrote this proof quite well. Some rough spots appeared regarding the logical order of the steps of the proof, including the transition from $|s_n - L| < \epsilon$ for all $n \geq N_1$ to $L - \epsilon < s_n < L + \epsilon$ for all $n \geq N_1$. It is important to understand that the needed inequalities are valid only when n is sufficiently large.

B: Proof B requires very careful language and logic. But properly organized it can be quite simple. If you feel that I misunderstood what you wrote, please explain it to me.

C: The idea here is to show that if $x_n \in [-M, M]$ for infinitely many values of $n \in \mathbb{N}$ then there is a subsequence x_{n_k} that lies in $[-M, M]$ for all $k \in \mathbb{N}$. Then apply the Bolzano-Weierstrass theorem to derive a contradiction to the hypothesis that x_n has no convergent subsequence.

Class Statistics

Grade	Test#1	Test#2	Test#3	Final Exam	Final Grade
90-100 (A)	1				
80-89 (B)	5				
70-79 (C)	2				
60-69 (D)	1				
0-59 (F)	1				
Test Avg	79.3%	%	%	%	%
HW Avg	7.8				
HW/Tst Correl	0.71				

The Correlation Coefficient is the cosine of the angle between two data vectors in $\mathbb{R}^{12}$–one dimension for each student enrolled. Thus this coefficient is between 1 and -1, with coefficients above 0.6 being considered strongly positive. The correlation coefficient shown indicates that the test grades in the course have a strongly positive correlation with performance on the homework.