Print Your Name Here: ______________________

- **Show all work:** *Answers without work are not sufficient.* We can give credit *only* for what you write! *Indicate clearly if you continue on the back side*, and *write your name at the top of the scratch sheet if you will turn it in for grading.*
- ***Books, notes (electronic or paper), cell phones, smart phones, and internet-connected devices are prohibited!*** A scientific calculator is allowed—but it is not needed. If you use a calculator, you must write out the operations performed on the calculator to show that you know how to solve the problem. *Please do not replace precise answers with decimal approximations.*
- There are **five (5)** problems: maximum total score = 100.

1. (20) Find the *general real-valued* solution to $y^{(4)} - 16y = 0$.

2. (20) Let $\{\phi_1, \phi_2, \phi_3\}$ be a set of solutions on $(-\infty, \infty)$ to the equation $y''' + a_1 y'' + a_2 y' + a_3 y = 0$ satisfying the following initial conditions: $\phi_1(0) = \phi_2'(0) = \phi_3''(0) = 1$ but $\phi_1'(0) = \phi_1''(0) = \phi_2(0) = \phi_2''(0) = \phi_3(0) = \phi_3'(0) = 0$.

a. Show that the set $\{\phi_1, \phi_2, \phi_3\}$ is linearly independent on $(-\infty, \infty)$.

b. Let y be the solution to the differential equation satisfying the initial conditions $y(0) = \alpha, y'(0) = \beta, y''(0) = \gamma$. Express y in terms of ϕ_1, ϕ_2 and ϕ_3.

3. (20) Find a linearly independent set $\{\phi_1, \phi_2, \phi_3\}$ of solutions to $y''' - 9y' = 0$ on the real line, and calculate the Wronskian $W(\phi_1, \phi_2, \phi_3)(x)$.

4. (20) For each statement, state whether it is **true or false**, and *explain* the reason for your answer.

a. The set $\{\phi_1, \phi_2, \phi_3\}$ of solutions to $y'' + a_1 y' + a_2 y = 0$ is a linearly independent set on $(-\infty, \infty)$.

b. If a set $\{\phi_1, \phi_2, \ldots, \phi_n\}$ of functions is linearly independent on $(0, 1)$ then it must be linearly independent on $(0, \infty)$.

5. (20) Find the *general solution* to the non-homogeneous equation $y'' + y = \cos x$. Suggestion: The differential operator $M = \dfrac{d^2}{dx^2} + 1$ annihilates $\cos x$. Use M to find a particular solution y_p to the non-homogeneous equation by setting $M(y'' + y) = 0$.

Solutions

1. $y = c_1e^{2x} + c_2e^{-2x} + c_3 \cos 2x + c_4 \sin 2x$ with real coefficients.

2.

a. You could do this by showing that the Wronskian at 0 is 1, or directly from the definition of linear independence.

b. $y = \alpha\phi_1 + \beta\phi_2 + \gamma\phi_3$ is the unique solution. (Remark: This exercise shows that the solution to an initial value problem is a linear function of the initial values.)

3. A basis for the solution space could be $\{1, e^{3x}, e^{-3x}\}$ and then the Wronskian would be ± 54 depending on the order in which you list the basis functions. Remark: Since the coefficient "a_1" of y'' is zero, the Wronskian is necessarily constant.

4.

a. The statement is **False** because the solution space for a second order linear homogeneous ordinary differential equation is only two *dimensional*. Thus any set with more than two solution functions must be linearly *dependent*. Remember that the solution set for the linear ODE has infinitely many functions in it. It is the *dimension* of the vector space that is critical here.

b. The statement is **True**, because if $c_1\phi_1(x) + \cdots + c_n\phi_n(x) = 0$ for all $x > 0$ then it must be zero for all $x \in (0, 1)$ and this implies that $c_1 = \cdots = c_n = 0$. Note that we are not given that these functions are solutions of any linear ordinary differential equation, or even that they are differentiable. So the Wronskian is irrelevant for this question. Also, remember the example $\{x^2, x|x|\}$ which is linearly *independent* on $(-\infty, \infty)$ although the Wronskian is identically zero.

5. We find that $y_p = \dfrac{x}{2} \sin x$, which is easily checked as follows: $y_p'' + y_p = \cos x$. Hence the *general solution* to the non-homogeneous equation is $y = c_1 \cos x + c_2 \sin x + \dfrac{x}{2} \sin x$. Notice that as $|x| \to \infty$ the values for $|y|$ can become as large as we like. This is a *resonance* phenomenon. This happens because the frequency of the *driving force* ($\cos x$) is the same as the natural frequency of the *non-forced* (homogeneous) system.

Class Statistics

% Grade	Test#1	Test#2	Test#3	Final Exam	Final Grade
90-100 (A)	3	8			
80-89 (B)	3	2			
70-79 (C)	2	1			
60-69 (D)	2	1			
0-59 (F)	2	0			
Test Avg	76.8%	89.2%	%	%	%
Cumulative HW Avg	87.1%	90.0%	%	%	%
HW/Test Correl	0.85	0.89			

The Correlation Coefficient is the cosine of the angle between two data vectors in $\mathbb{R}^{12}$– one dimension for each student enrolled. Thus this coefficient is between 1 and -1, with coefficients above 0.6 being considered strongly positive. The correlation coefficient shown indicates that the test grades in the course have a very strongly positive correlation with performance on the homework.