Print Your Name Here: ______________________________

- **Show all work:** *Answers without work are not sufficient.* We can give credit *only* for what you write! *Indicate clearly if you continue on the back side*, and *write your name at the top of the scratch sheet if you will turn it in for grading.*
- ***Books, notes (electronic or paper), cell phones, smart phones, and internet-connected devices are prohibited!*** A scientific calculator is allowed—but it is not needed. If you use a calculator, you must write out the operations performed on the calculator to show that you know how to solve the problem. *Please do not replace precise answers with decimal approximations.*
- There are **FIVE (5)** problems: maximum total score = 100.

1. (20) Consider the equation $y' + \frac{2}{x}y = \frac{1}{x^2}$ with $x > 0$.

a. (15) Use an *integrating factor* to find the *general solution*.

b. (5) Find the solution for which $y(1) = 2$.

2. (20) Find the general solution to $y'' - 4y' + 4y = 0$.

3. (20) Solve the initial value problem: $y'' + y = 0,\ y\left(\frac{\pi}{2}\right) = i,\ y'\left(\frac{\pi}{2}\right) = -3.$

4. (30) Explain why each pair is either linearly *independent* or linearly *dependent* on the given interval.

a. x^2 and $x|x|$ on $[-1, 1]$.

b. $\sin x$ and $(e^{ix} - e^{-ix})$ on $(-\infty, \infty)$.

c. e^{3x} and xe^{3x} on $(-\infty, \infty)$.

5. (10) Suppose $x\phi'(x) + \phi(x) \leq 1$ if $0 \leq x < \infty$. Prove that $\phi(x) \leq 1$ on $[0, \infty)$.

Solutions

1.

a. The integrating factor is $e^{2\ln x} = x^2$ and the general solution is $y = \dfrac{1}{x} + \dfrac{C}{x^2}$, $x > 0$. The greatest difficulty was either not knowing the anti-derivative of $\dfrac{2}{x}$ or not knowing that $e^{2\ln x} = x^2$. For a course in differential equations one must be comfortable with basic calculus. Working on the practice exercises is a good way to review your calculus in case it has been a few years since your previous mathematics courses.

b. $y = \dfrac{1}{x} + \dfrac{1}{x^2}$.

2. The characteristic polynomial is $r^2 - 4r + 4 = (r-2)^2 = 0$, which has a double root. The general solution is $y = e^{2x}(C_1 + C_2 x)$.

3. One needs to find the general solution first, in order to solve the initial value problem that is given. The characteristic polynomial is $r^2 + 1 = 0$, so that $r = \pm i$. Using Euler's formula, $\cos x$ and $\sin x$ comprise a linearly independent pair of real-valued solutions to the original equation with real coefficients. Alternatively, one could choose to use the complex solutions e^{ix} and e^{-ix}. The general solution could be expressed as $y = C_1 \cos x + C_2 \sin x$ or else as $y = C_1 e^{ix} + C_2 e^{-ix}$. The solution for C_1 and C_2 in the initial value problem will depend on which choice you made to describe the general solution. The solution to the initial value problem can be written in either of two ways: $y = 3\cos x + i \sin x$ or $y = 2e^{ix} + e^{-ix}$. Either answer would be correct. The coefficients are found, after picking either the real or complex functions, by finding y and y' at $\dfrac{\pi}{2}$. If you made the complex choice, of course you need to know (from Euler's formula) that $e^{i\frac{\pi}{2}} = i$.

4.

a. Suppose $C_1 x^2 + C_2 x|x| = 0$ on $[-1, 1]$. Substituting $x = \pm 1$ we see that $C_1 + C_2 = 0$ and $C_1 - C_2 = 0$. It follows that both constants are 0, proving linear independence on $[-1, 1]$.

b. $2i \sin x - (e^{ix} - e^{-ix}) \equiv 0$ on $(-\infty, \infty)$, proving linear dependence.

c. The functions e^{3x} and xe^{3x} on $(-\infty, \infty)$ comprise a basis for the solution space of $y'' - 6y' + 9y = 0$ on the real line, making the pair linearly independent there. Alternatively, $(C_1 + C_2 x)e^{3x} \equiv 0$ on the real line forces $C_1 + C_2 x \equiv 0$ on $(-\infty, \infty)$. This forces $C_1 = 0 = C_2$, establishing linear independence, since a nontrivial linear polynomial has at most one root.

5. If we substitute $x = 0$ we see at once that $\phi(0) \leq 1$. Now suppose $x > 0$. The key is that the left side of the given inequality is $(x\phi(x))'$. Then $\displaystyle\int_0^x \frac{d}{dt}(t\phi(t))\, dt \leq \int_0^x 1\, dt = x$. Thus $x\phi(x) \leq x$, so that $\phi(x) \leq 1$ for all $x \geq 0$. It is important to use a definite integral here because anti-differentiation does not preserve inequalities. (This exercise is modeled on the proof of our existence and uniqueness theorem.)

Class Statistics

% Grade	Test#1	Test#2	Test#3	Final Exam	Final Grade
90-100 (A)	3				
80-89 (B)	3				
70-79 (C)	2				
60-69 (D)	2				
0-59 (F)	2				
Test Avg	76.8%	%	%	%	%
Cumulative HW Avg	87.1%	%	%	%	%
HW/Test Correl	0.85				

The Correlation Coefficient is the cosine of the angle between two data vectors in $\mathbb{R}^{12}$–one dimension for each student enrolled. Thus this coefficient is between 1 and -1, with coefficients above 0.6 being considered strongly positive. The correlation coefficient shown indicates that the test grades in the course have a strongly positive correlation with performance on the homework.