Print Your Name Here: ______________________________

- **Show all work** in the space provided. We can give credit *only* for what you write! *Indicate clearly if you continue on the back side.*
- **No** books or notes (paper or electronic) or communication devices (smart/cell phones, internet-connected devices such as laptops, tablets, or I-watches) are allowed. A scientific calculator (*not capable* of graphing or symbolic calculations) is allowed—but it is not needed. If you use a calculator, you *must still write out all operations performed* on the calculator. ***Do not*** replace precise answers such as $\sqrt{2}$, $\frac{1}{3}$, or π with decimal approximations. *Keep your eyes on your own paper!*
- The maximum score possible is 200 points.

1. (20 points) Show that the following limit does *not* exist, by evaluating along 2 suitable paths and obtaining different results: $\lim_{(x,y)\to(0,0)} \frac{(x+y)^2}{x^2+y^2}$.

2. (20 points) Let $F(x,y,z) = x^2 + 2y^2 + 3z^2$.

a. Find the gradient vector ∇F as a function of x, y, and z.

b. Find an equation for the tangent plane to the ellipsoid $F(x,y,z) = 6$ at the point $(1,1,1)$.

3. (20 points) *Find the point* (x, y, z) on the plane $x + 2y + 3z = 6$ that *minimizes* the value of $f(x, y, z) = (x^2 + y^2 + z^2)$. Lagrange multipliers will be the simplest method.

4. (20 points) Evaluate $\iint_D x \, dA$ if D is the region bounded by the curves $y = x^2$ and $x = y^2$.

5. (20 points) Find the surface area of that part of the surface $z = xy$ that lies inside the cylinder $x^2 + y^2 = 1$.

6. (20 points) Find $\iiint\limits_T y\,dV$ where T is the tetrahedron with vertices $(0,0,0), (2,0,0), (0,2,0)$ and $(0,0,2)$.

7. (20 points) Use *spherical coordinates* to find the volume of that part of the *ball* $\rho \leqslant 1$ that lies *between* the two cones $\phi = \frac{\pi}{6}$ and $\phi = \frac{\pi}{3}$.

8. (20 points) Use *partial derivatives* to check that the vector field $\vec{F} = \langle 2xy,\, x^2 + 2yz,\, y^2 \rangle$ is *conservative*. Then find a *potential function* $f(x,y,z)$ for $\vec{F}(x,y,z)$, and use $f(x,y,z)$ to *evaluate* $\displaystyle\int_{(0,0,0)}^{(1,2,3)} 2xy\,dx + (x^2 + 2yz)\,dy + y^2\,dz$.

9. (20 points) Let S be the top and 4 sides (but *not* the bottom) of the unit cube $B = \{(x, y, z) \mid 0 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant 1,\ 0 \leqslant z \leqslant 1\}$. Give S the outward unit normal $\mathbf{n}$ for orientation, and let C be the the positively oriented boundary curve of S, which makes C a square in the xy-plane. If the vector field $\mathbf{F}(x, y, z) = \langle xyz,\ xy,\ x^2yz\rangle$, *use Stokes' Theorem* to evaluate $\iint_S (\nabla \times\ \mathbf{F}) \cdot\ \mathbf{n}\, dS$.

10. (20 points) Let S be the surface of the region R in the *first octant* bounded by the paraboloid $z = 1 - x^2 - y^2$ and by the three coordinate planes. Give S the outward unit normal $\mathbf{n}$. If $\mathbf{F}(x, y, z) = \langle z,\ x,\ zx\rangle$, use the Divergence Theorem (Gauss's Theorem) to evaluate $\iint_S \mathbf{F} \cdot \mathbf{n}\, dS$.

Solutions

1. Along $y = -x$ the limit is 0, but along $y = x$ the limit is 2, for example. There are other valid choices. But remember that your chosen paths *must* go through the origin! For example, you would not take I10 to go to Montreal. See 14.2/15.

2. See 14.6/47.

- **a**. $\nabla F(x, y, z) = \langle 2x, 4y, 6z \rangle$.
- **b**. $\nabla F(1, 1, 1) \cdot \langle x - 1, y - 1, z - 1, \rangle = 0$, or $x + 2y + 3z = 6$. Remember that the equation of a plane *must* be *linear*! It is important to understand the geometrical content of this question: The normal vector being perpendicular to the plane means that the dot product of the gradient at the indicated point with a vector in the plane *must* equal *zero*.

3. Let $g(x, y, z) = x + 2y + 3z$ and set $\nabla f = \lambda \nabla g$. It follows that $y = 2x$ and $z = 3x$. Thus $x + 2y + 3z = 14x = 6$. Thus $(x, y, z) = \left(\frac{3}{7}, \frac{6}{7}, \frac{9}{7}\right)$. Compare with 14.8/9.

4. $\displaystyle\iint\limits_D x\, dA = \int_0^1 \int_{x^2}^{\sqrt{x}} x\, dy\, dx = \int_0^1 x(\sqrt{x} - x^2)\, dx = \frac{3}{20}$. The most common error was to reverse the upper and lower endpoints in the inner integral. But one should realize that the integral over a region with positive area of a function that is positive must be positive. See 15.2/7.

5. $\displaystyle A = \iint\limits_{x^2+y^2 \leqslant 1} \sqrt{1 + x^2 + y^2}\, dA = \int_0^{2\pi} \int_0^1 \sqrt{1 + r^2}\; r\, dr\, d\theta = \frac{2\pi}{3}(2\sqrt{2} - 1)$. See 15.5/11.

6. $\displaystyle\iiint\limits_T y\, dV = \int_0^2 \int_0^{2-x} \int_0^{2-x-y} y\, dz\, dy\, dx = \frac{2}{3}$. It is helpful *not* to expand the powers of $2 - x$. Remember that a *factored* polynomial is much *finer* than one that is *multiplied* out, making a needless *mess*. Also, please remember that the integral of a positive function over a domain of positive volume cannot be negative! Also, it is very important to be able to distinguish between a tetrahedron and a cube! See 15.6/19.

7. $\displaystyle V = \int_0^{2\pi} \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \int_0^1 \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta = \frac{\pi}{3}(\sqrt{3} - 1)$. See 15.8/29.

8. Check that $\nabla \times \mathbf{F} = \mathbf{0}$. Since $\mathbb{R}^3$ is simply connected, this proves that $\mathbf{F}$ is conservative. After doing that, since $f_x = 2xy$, $f = x^2y + \phi(y, z)$ alone. Since $f_y = x^2 + +\phi_y = x^2 + 2yz$, $\phi = y^2z + \psi(z)$ alone. Finally, since $f_z = y^2 + \psi_z = y^2$, $\psi(z)$ is an arbitrary constant, which you may as well choose to be zero. Thus $f(x, y, z) = x^2y + y^2z$ and the integral $\displaystyle\int_{(0,0,0)}^{(1,2,3)} 2xy\, dx + (x^2 + 2yz)\, dy + y^2\, dz =$ $f(1, 2, 3) - f(0, 0, 0) = 14$. On many papers, the method for some or all of the three parts was unclear or muddled. See 16.3/21.

9. $\displaystyle\iint\limits_S (\nabla \times\ \mathbf{F}) \cdot\ \mathbf{n}\, dS = \oint\limits_C xyz\, dx + xy\, dy + x^2yz\, dz = \oint\limits_C xy\, dy = \int_0^1 \int_0^1 y\, dy\, dx = \frac{1}{2}$, where we have noted that $z = 0$ everywhere on the curve C, and we used Green's Theorem to arrive at the final double integral. Many students ignored the instruction to use Stokes' theorem and actually calculated the curl. Some others confused Stokes' theorem with the Divergence theorem, which does not apply to a surface that does not enclose a region. See 16.8/5.

10. $\displaystyle\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iiint_R x\,dV = \int_0^{\frac{\pi}{2}}\int_0^1\int_0^{1-r^2}(r\cos\theta)r\,dz\,dr\,d\theta = \frac{2}{15}$. It is important to apply the Divergence theorem correctly, and it is advisable to use cylindrical coordinates for the integral over the enclosed region R in the first octant. Spherical coordinates are not appropriate for the given region R because of the paraboloid providing the top and side surface. See 16.9/15,11.

Class Statistics

% Grade	Test#1	Test#2	Test#3	Final Exam	Final Grade
90-100 (A)	13	16	14	11	13
80-89 (B)	6	7	6	13	10
70-79 (C)	6	4	5	3	6
60-69 (D)	5	1	1	3	2
0-59 (F)	0	2	4	1	0
This Test Avg	84.8%	86.0%	81.6%	84.5%	86.9%
HW Avg	87.5%	88.1%	88.4%	88.4%	88.4%
HW/TST Corr	0.61	0.77	0.70	0.39	0.39

The Correlation Coefficient is the cosine of the angle between two data vectors in $\mathbb{R}^{31}$–one dimension for each student enrolled. Thus this coefficient is between 1 and -1, with coefficients above 0.6 being considered strongly positive. The correlation coefficient shown indicates that the test grades in the course have a positive correlation with performance on the homework.