Matroids Arising From Nested Sequences of Flats In Projective And Affine Geometries

Matthew Mizell
Mathematics Department
Louisiana State University
Baton Rouge, Louisiana, USA
mmizel4@lsu.edu

James Oxley
Mathematics Department
Louisiana State University
Baton Rouge, Louisiana, USA
oxley@math.lsu.edu

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Abstract

Targets are matroids that arise from a nested sequence of flats in a projective geometry. This class of matroids was introduced by Nelson and Nomoto, who found the forbidden induced restrictions for binary targets. This paper generalizes their result to targets arising from projective geometries over $GF(q)$. We also consider targets arising from nested sequences of affine flats and determine the forbidden induced restrictions for affine targets.

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1 Introduction

Throughout this paper, we follow the notation and terminology of [3]. All matroids considered here are simple. This means, for example, that when we contract an element, we always simplify the result. An *induced restriction* of a matroid M is a restriction of M to one of its flats.

Let M be a rank-r projective or affine geometry represented over $GF(q)$. We call $(F_0, F_1, \dots, F_k)$ a *nested sequence of projective flats* or a *nested sequence of affine flats* if $\emptyset = F_0 \subseteq F_1 \subseteq \cdots \subseteq F_{k-1} \subseteq F_k = E(M)$ and each F_i is a, possibly empty, flat of M. Let (G, R) be a partition of $E(M)$ into, possibly empty, subsets G and R. We call the elements in G *green*; those in R are *red*. A subset X of $E(M)$ is *monochromatic* if $X \subseteq G$ or $X \subseteq R$. For a subset X of $E(PG(r-1,q))$, we call $PG(r-1,q)|X$ a *projective target*, or a *target*, if there is a nested sequence $(F_0, F_1, \dots, F_k)$ of projective flats such that X is the union of all sets $F_{i+1} - F_i$ for i even. It is straightforward to check that $PG(r-1,q)|G$ is a target if and only if $PG(r-1,q)|R$ is a target. Because $GF(q)$-representable matroids are not necessarily uniquely $GF(q)$-representable, we have defined targets in terms of 2-colorings of $PG(r-1,q)$. When $X \subseteq E(AG(r-1,q))$, we call $AG(r-1,q)|X$ an *affine*

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target if there is a nested sequence $(F_0, F_1, \dots, F_k)$ of affine flats such that X is the union of all sets $F_{i+1} - F_i$ for i even. For affine targets in $AG(r-1,q)$, we follow the same convention of defining targets in terms of 2-colorings.

Consider an analogous construction for graphs, that is, take a sequence $(K_0, K_1, \dots, K_n)$ of complete graphs where K_{i+1} has K_i as a subgraph for each i in $\{1, 2, \dots, n-1\}$. Moreover, for each such i, color the vertex v of $V(K_{i+1}) - V(K_i)$ either green or red and color all the edges of $E(K_{i+1}) - E(K_i)$ the same color as v. This process is equivalent to repeatedly adding green or red dominating vertices, that is, adding a green or red vertex v to a graph G that is adjacent to every vertex u in $V(G-v)$. Consider the subgraph H of K_n whose vertex set is $V(K_n)$ and whose edge set is the set of green edges. Observe that, in the construction of H, when a red dominating vertex is added, it is an isolated vertex of the graph that has been constructed so far. Therefore, to construct H, at each step, we are adding a green dominating vertex or a red isolated vertex. Chvátal and Hammer [1] showed that the class of graphs that arises from repeatedly adding dominating vertices and isolated vertices coincides with the class of threshold graphs. This is the class of graphs that has no induced subgraph that is isomorphic to $C_4, 2K_2$, or P_4, that is a 4-cycle, two non-adjacent edges, or a 4-vertex path.

Nelson and Nomoto [2] introduced binary targets and characterized them as follows.

Theorem 1. *Let (G,R) be a 2-coloring of $PG(r-1,2)$. Then $PG(r-1,2)|G$ is a target if and only if it does not contain $U_{3,3}$ or $U_{2,3} \oplus U_{1,1}$ as an induced restriction.*

Nelson and Nomoto [2] call $U_{3,3}$ the *claw*, while they call $U_{2,3} \oplus U_{1,1}$, the complement of $U_{3,3}$ in F_7, the *anti-claw*. They derive Theorem 1 as a consequence of a structural description of claw-free binary matroids. In Section 3, we give a proof of that theorem that does not rely on this structural description. Then, for all $q \geqslant 3$, we characterize targets represented over $GF(q)$ in terms of forbidden induced restrictions by proving the next result.

Theorem 2. *For a prime power q exceeding two, let (G,R) be a 2-coloring of $PG(r-1,q)$. Then $PG(r-1,q)|G$ is a target if and only if it does not contain any of $U_{2,2}$, $U_{2,3}, \dots, U_{2,q-2}$, or $U_{2,q-1}$ as an induced restriction.*

In Section 2, we prove some useful properties of targets. In particular, we show that targets are closed under contractions. A simple matroid N is an *induced minor* of a simple matroid M if N can be obtained from M by a sequence of contractions and induced restrictions. We observe that Theorem 2 can also be viewed as characterizing targets in terms of forbidden induced minors. Our other main theorems, which are proved in Section 4, characterize affine targets in terms of forbidden induced restrictions. There are three cases, depending on the value of q.

Theorem 3. *Let (G,R) be a 2-coloring of $AG(r-1,2)$. Then $AG(r-1,2)|G$ is an affine target if and only if it does not contain $U_{4,4}$ as an induced restriction.*

The matroids $\mathcal{W}^3$ and $P(U_{2,3}, U_{2,3})$ that appear in the next theorem are the rank-3 whirl and the parallel connection of two copies of $U_{2,3}$.

Theorem 4. *Let (G, R) be a 2-coloring of $AG(r-1,3)$. Then $AG(r-1,3)|G$ is an affine target if and only if it does not contain any of $U_{3,3}, U_{3,4}, U_{2,3} \oplus U_{1,1}, U_{2,3} \oplus_2 U_{2,4}, P(U_{2,3}, U_{2,3})$, or $\mathcal{W}^3$ as an induced restriction.*

Theorem 5. *Let (G, R) be a 2-coloring of $AG(r-1,q)$, for $q \geqslant 4$. Then $AG(r-1,q)|G$ is an affine target if and only if it does not contain any of $U_{2,2}, U_{2,3}, \dots, U_{2,q-3}$, or $U_{2,q-2}$ as an induced restriction.*

2 Preliminary Results

Throughout the paper, we will refer to flats and hyperplanes of $PG(r-1,q)$ as *projective flats* and *projective hyperplanes*, respectively. Let M be a restriction of $PG(r-1,q)$. For a subset X of $E(M)$, its *projective closure*, $\mathrm{cl}_P(X)$, is the closure of X in the matroid $PG(r-1,q)$. We first show that if $PG(r-1,q)|G$ is a target, then the matroid $PG(r-1,q)|G$ is uniquely determined by the sequence $(r_0, r_1, \dots, r_k)$ of ranks of the nested sequence $(F_0, F_1, \dots, F_k)$ of projective flats. Note that we shall often write G and R for the matroids $PG(r-1,q)|G$ and $PG(r-1,q)|R$, respectively. This means that we will be using G and R to denote both matroids and the ground sets of those matroids.

Proposition 6. *Let $(E_0, E_1, \dots, E_k)$ and $(F_0, F_1, \dots, F_k)$ be nested sequences of flats in $PG(r-1,q)$ such that $r(E_i) = r(F_i)$ for all i in $\{0, 1, \dots, k-1\}$. Let G_E and G_F be the union, respectively, of all $E_{i+1} - E_i$ and of all $F_{i+1} - F_i$ for the even numbers i in $\{0, 1, \dots, k-1\}$. Then $PG(r-1,q)|G_E \cong PG(r-1,q)|G_F$.*

Proof. Let h be the smallest i such that $r(E_i) > 0$. Let $\{b_{h,1}, b_{h,2}, \dots, b_{h,m_h}\}$ and $\{d_{h,1}, d_{h,2}, \dots, d_{h,m_h}\}$ be bases B_h and D_h of $PG(r-1,q)|E_h$ and $PG(r-1,q)|F_h$, respectively. Let $B_0 = B_1 = \dots = B_{h-1} = \emptyset$ and $D_0 = D_1 = \dots = D_{h-1} = \emptyset$. For $j \geqslant h$, assume that $B_0, B_1, \dots, B_j$ and $D_0, D_1, \dots, D_j$ have been defined. Let B_{j+1} and D_{j+1} be bases of E_{j+1} and F_{j+1}, respectively, such that $B_j \subseteq B_{j+1}$ and $D_j \subseteq D_{j+1}$. Let $B_{j+1} - B_j = \{b_{j+1,1}, b_{j+1,2}, \dots, b_{j+1,m_{j+1}}\}$ and $D_{j+1} - D_j = \{d_{j+1,1}, d_{j+1,2}, \dots, d_{j+1,m_{j+1}}\}$. Define the automorphism ϕ on $PG(r-1,q)$ by $\phi(b_{s,t}) = d_{s,t}$ for all s and t such that $s \geqslant h$. Then $\phi(E_i) = F_i$ for all i, so $\phi(E_{i+1} - E_i) = \phi(E_{i+1}) - \phi(E_i) = F_{i+1} - F_i$, for all i. Therefore, $PG(r-1,q)|G_E \cong PG(r-1,q)|G_F$. □

The last result means that we can refer to a simple $GF(q)$-representable matroid M as being a target exactly when some, and hence all, of the $GF(q)$-representations of M are targets. Note that in a nested sequence $(F_0, F_1, \dots, F_k)$ of flats defining a target, it is convenient to allow equality of the flats. A nested sequence $(F_0, F_1, \dots, F_k)$ of flats is the *canonical nested sequence* defining a projective or affine target if $F_0 = \emptyset$, and $F_1, F_2, \dots, F_{k-1}$, and F_k are distinct. Observe that allowing F_1 to be empty accommodates the requirement that the target is the union of all sets $F_{i+1} - F_i$ for i even.

Lemma 2.15 of Nelson and Nomoto [2] proved that binary targets are closed under induced restriction. Using the same proof, their result can be extended to targets represented over $GF(q)$.

Lemma 7. *The class of targets over $GF(q)$ is closed under induced restrictions.*

Lemma 8. *Let (G, R) be a 2-coloring of $PG(r-1, q)$. Assume that G is a target and F is a projective flat of $PG(r-1, q)$. Then exactly one of $G \cap F$ and $R \cap F$ has rank $r(F)$.*

Proof. By Lemma 7, $PG(r-1, q)|(G \cap F)$ is a target corresponding to a nested sequence $(F'_0, F'_1, \ldots, F'_{k-1}, F)$ of projective flats. By, for example, [4, Lemma 2.1], $r(G \cap F)$ or $r(R \cap F)$ is $r(F)$. Either $G \cap F$ or $R \cap F$ is contained in some proper projective flat of F. Therefore, either $r(G \cap F) < r(F)$ or $r(R \cap F) < r(F)$. □

We refer to the rank of the set of green elements in a projective flat F as the *green rank* of F. If F has green rank $r(F)$, we say that F is a *green flat*. Furthermore, if a projective hyperplane has green rank $r-1$, then it is a *green hyperplane*. Red rank, red flats, and red hyperplanes are defined analogously. From the last lemma, it follows that a projective flat can either be a green flat or a red flat, but not both.

We now show that every contraction of a target is a target. Consider contracting a green element e in M. If a parallel class in the contraction contains at least one green point, then, after the simplification, the resulting point will be green. If there are only red points in the parallel class, then, after the simplification, the resulting point is red.

Proposition 9. *The class of targets over $GF(q)$ is closed under contractions.*

Proof. Let (G, R) be a 2-coloring of $PG(r-1, q)$. Assume that G is a target. Then there is a canonical nested sequence $(F_0, F_1, \ldots, F_k)$ of projective flats such that G is the union of all sets $F_{i+1} - F_i$ for i even. Let e be an element of $F_m - F_{m-1}$ where F_m is a green flat. Then the elements of $F_m - F_{m-1}$ are green. Suppose x is a red point in F_m. Then $x \in F_{m-1}$. If $y \in \mathrm{cl}_P(\{e, x\})$, then $y \notin F_{m-1}$, otherwise the circuit $\{e, x, y\}$ gives the contradiction that e is an element of F_{m-1}. Since $\{e, x\} \subseteq F_m$, we must have that y is in F_m, so y is in $F_m - F_{m-1}$. Hence y is green. We deduce that, in the contraction of e, every element of $F_m - e$ is green.

Now assume F_j is a red flat containing F_m. Then $F_j - F_{j-1} \subseteq R$. Consider a point z in $F_j - F_{j-1}$. Using a symmetric argument to that given above, we deduce that e is the only point of $\mathrm{cl}_P(\{e, z\})$ not in $F_j - F_{j-1}$. Therefore, the points in $(F_j - F_{j-1}) - e$ are red. Clearly, if F_k is a green flat containing F_m, then the points in $(F_k - F_{k-1}) - e$ are green. Thus, in $\mathrm{si}(PG(r-1, q)/e)$, we have $(\mathrm{si}(F_m - e), \mathrm{si}(F_{m+1} - e), \ldots, \mathrm{si}(F_k - e))$ as a nested sequence of projective flats. Writing this new nested sequence of projective flats in $PG(r-2, q)$ as $(F'_m, F'_{m+1}, \ldots, F'_k)$, we see that F'_m is entirely green and, for each $i \geqslant 1$, the set $F'_{m+i} - F'_{m+i-1}$ is entirely red if i is odd and is entirely green if i is even. Hence $\mathrm{si}(G/e)$ is a target. □

Combining Lemma 7 and Proposition 9, we get the following.

Corollary 10. *The class of targets over $GF(q)$ is closed under induced minors.*

Lemma 11. *Let (G, R) be a 2-coloring of $PG(r-1, q)$. If G is a target, then G and R are connected unless $q = 2$ and G or R is $U_{2,2}$.*

Proof. Assume that the exceptional case does not arise and that $r(G) \geqslant r(R)$. If $G = PG(r-1,q)$, then the result holds. Assume G is not the whole projective geometry. Then G contains $AG(r-1,q)$, so G is connected. Similarly, R will also have an affine geometry as a restriction. Thus R is certainly connected when $r(R) = r(G)$. Assume $r(R) < r(G)$. Take a projective flat F that has R as a spanning restriction. Then $r(R) \geqslant r(G \cap F)$ so, as above, we deduce that R is connected. $\square$

If (G, R) is a 2-coloring of $PG(r-1,q)$, then G is a *minimal non-target* if G is not a target but every proper induced restriction of G is a target. Clearly, if G is a minimal non-target, then R is not a target. But if $r(R) > r(G)$, then R is not a minimal non-target.

Lemma 12. *Let (G, R) be a 2-coloring of $PG(r-1,q)$. Suppose $PG(r-1,q)|G$ is a minimal non-target of rank r. Then $r(R) = r$.*

Proof. Assume $r(R) < r$. Then there is a hyperplane H containing R. Since $PG(r-1,q)|(G \cap H)$ is a target, R is a target. However, this implies that G is a target, a contradiction. Therefore $r(R) = r$. $\square$

3 Forbidden Induced Restrictions of Target Matroids

This section contains the proofs of Theorem 1 and Theorem 2.

Proof of Theorem 1. Assume that G is a target. If there is a projective flat F such that $PG(r-1,2)|(G \cap F) \cong U_{3,3}$, then $PG(r-1,2)|(R \cap F) \cong U_{2,3} \oplus U_{1,1}$. Since $PG(r-1,2)|(G \cap F)$ is a target, this contradicts Lemma 8, as $r(G \cap F) = r(R \cap F)$.

Let (G, R) be a 2-coloring of $PG(r-1,2)$. Suppose that G is a rank-r minimal non-target. Assume that G does not have $U_{3,3}$ or $U_{2,3} \oplus U_{1,1}$ as an induced restriction. Thus $r \geqslant 4$. By Lemma 12, $r(G) = r(R) = r$. Then there are a red hyperplane H_0 and a green hyperplane H_1. Let $F = H_0 \cap H_1$. Then $r(F) = r-2$. We may assume that F is green. Thus all of the points in $H_0 - F$ are red. Let H_2 be the other hyperplane containing F. As $r(G) = r$, each of $H_1 - F$ and $H_2 - F$ contain a green point. Since $r(R) = r$, there is a red point z in $H_1 - F$ or $H_2 - F$. We may assume that $z \in H_1 - F$. Consider the set W of points w in $H_1 - F$ such that the line $\mathrm{cl}_P(\{z, w\})$ contains a green point in F. As H_1 is green, there is a green element w_1 in W. Let g be a green element in $H_2 - F$. Now consider the plane $P = \mathrm{cl}_P(\{g, w_1, z\})$. As $w_1 \in W$, the third point x on the line $\mathrm{cl}_P(\{w_1, z\})$ is a green point in F. Now consider the line $\mathrm{cl}_P(\{g, z\})$. The third point y_0 on this line is in $H_0 - F$, so y_0 is red. Finally, the third point y_1 on the line $\mathrm{cl}_P(\{x, y_0\})$ is also in $H_0 - F$, so y_1 is red. Clearly, the points x, y_0, and y_1 are elements of P and $r(R \cap \{y_0, y_1, z\}) = 3$ and $r(G \cap \{g, w_1, x\}) = 3$. Therefore, $r(G \cap P) = r(R \cap P) = 3$, a contradiction to Lemma 8. $\square$

Proof of Theorem 2. Assume that G is a target. If there is a projective flat F such that $PG(r-1,q)|(G \cap F)$ is any of $U_{2,2}, U_{2,3}, \dots, U_{2,q-2}$, or $U_{2,q-1}$, then, letting $F' = \mathrm{cl}_P(G \cap F)$, we have $r(G \cap F') = r(R \cap F')$, a contradiction to Lemma 8.

Let (G, R) be a 2-coloring of $PG(r-1, q)$. Suppose that G is a rank-r minimal non-target. Assume that G does not have $U_{2,2}, U_{2,3}, \dots, U_{2,q-2}$, or $U_{2,q-1}$ as an induced restriction. Thus $r \geqslant 3$. As in the last proof, $r(G) = r(R) = r$ and there are a red hyperplane H_0 and a green hyperplane H_1. Moreover, we may assume that their intersection F is green, so $H_0 - F$ is monochromatic red. Let $H_2, H_3, \dots, H_{q-1}$, and H_q be the other hyperplanes containing F. As $r(G) = r$, there are green points x and y such that $r(G \cap (F \cup \{x, y\})) = r$. Consider the projective line L that is spanned by x and y. Then L intersects each of $H_0 - F, H_1 - F, \dots, H_{q-1} - F$, and $H_q - F$ at a point. Since $H_0 - F$ is monochromatic red, L has a single red point in $H_0 - F$. Then there is a green point in each of $H_1 - F, H_2 - F, \dots, H_{q-1} - F$, and $H_q - F$. Therefore, as F is green, each of $H_1, H_2, \dots, H_{q-1}$, and H_q are also green. Since $r(R) = r$, there is a red point z outside H_0. We may assume that $z \in H_1$. For each point e in $H_0 - F$, the line $\mathrm{cl}_P(\{e, z\})$ is red, and therefore has at most one green point. Moreover, this line meets each of $H_3 - F$ and $H_4 - F$. Considering all choices of e, we see that at least half of the points in $(H_3 \cup H_4) - F$ are red. We may assume that at least half of the points in $H_3 - F$ are red. Observe that $PG(r-1, q)|(H_3 - F) \cong AG(r-2, q)$, so $|H_3 - F| = q^{r-2}$. Then $|R \cap (H_3 - F)| \geqslant \frac{1}{2}q^{r-2} > q^{r-3}$. Therefore, no hyperplane of $PG(r-1, q)|H_3$ contains $R \cap (H_3 - F)$. Hence $r(R \cap H_3) = r - 1$. This contradicts the fact that H_3 is green. □

4 Affine Target Matroids

In this section, we look at targets arising from affine geometries. This section begins with preliminary results about affine targets and minimal affine-non-targets. It concludes with the forbidden induced restrictions for affine targets over $GF(q)$. One fact that we use repeatedly is that if (G, R) is a 2-coloring of $AG(r-1, q)$, then G is an affine target if and only if R is an affine target. Viewing $AG(r-1, q)$ as a restriction, $PG(r-1, q)|X$, of $PG(r-1, q)$ obtained by deleting a projective hyperplane H from $PG(r-1, q)$, we call H the *complementary hyperplane of* X. We shall also refer to H as the *complementary hyperplane of* $AG(r-1, q)$.

Proposition 13. *Let* $(E_0, E_1, \dots, E_k)$ *and* $(F_0, F_1, \dots, F_k)$ *be nested sequences of flats in* $AG(r-1, q)$ *such that* $r(E_i) = r(F_i)$ *for all* i *in* $\{0, 1, \dots, k-1\}$. *Let* H *and* H' *be the complementary hyperplanes of* E_k *and* F_k, *respectively. Let* G_E *and* G_F *be the union, respectively, of all* $E_{i+1} - E_i$ *and of all* $F_{i+1} - F_i$ *for the even numbers* i *in* $\{0, 1, \dots, k-1\}$. *Then* $AG(r-1, q)|G_E \cong AG(r-1, q)|G_F$.

Proof. Observe that $E_k = E(PG(r-1, q)) - H$ and $F_k = E(PG(r-1, q)) - H'$. Let h be the smallest i such that $r(E_i) > 0$. Let $\{b_{h,1}, b_{h,2}, \dots, b_{h,m_h}\}$ and $\{d_{h,1}, d_{h,2}, \dots, d_{h,m_h}\}$ be bases B_h and D_h of $PG(r-1, q)|(\mathrm{cl}_P(E_h) - E_h)$ and $PG(r-1, q)|(\mathrm{cl}_P(F_h) - F_h)$, respectively. Let v and v' be elements in E_h and F_h, respectively. Then $\{v, b_{h,1}, b_{h,2}, \dots, b_{h,m_h}\}$ is a basis for $PG(r-1, q)|\,\mathrm{cl}_P(E_h)$ and $\{v', d_{h,1}, d_{h,2}, \dots, d_{h,m_h}\}$ is a basis for $PG(r-1, q)|\,\mathrm{cl}_P(F_h)$. Let $B_0 = B_1 = \dots = B_{h-1} = \emptyset$ and $D_0 = D_1 = \dots = D_{h-1} = \emptyset$. For $j \geqslant h$, assume that $B_0, B_1, \dots, B_j$ and $D_0, D_1, \dots, D_j$ have been defined. Let B_{j+1} and D_{j+1} be bases of $PG(r-1, q)|(\mathrm{cl}_P(E_{j+1}) - E_{j+1})$ and $PG(r-1, q)|(\mathrm{cl}_P(F_{j+1}) - F_{j+1})$, respectively, such that $B_j \subseteq B_{j+1}$ and $D_j \subseteq D_{j+1}$. Observe that adding v and v' to B_{j+1} and D_{j+1}, respectively, gives bases for $PG(r-1,q)|\operatorname{cl}_P(E_{j+1})$ and $PG(r-1,q)|\operatorname{cl}_P(F_{j+1})$ for all j. Let $B_{j+1} - B_j = \{b_{j+1,1}, b_{j+1,2}, \dots, b_{j+1,m_{j+1}}\}$ and $D_{j+1} - D_j = \{d_{j+1,1}, d_{j+1,2}, \dots, d_{j+1,m_{j+1}}\}$. Observe that B_k and D_k are bases for H and H', respectively. Now, $G_E = \operatorname{cl}_P(G_E) - H$ and $G_F = \operatorname{cl}_P(G_F) - H'$. Define the automorphism ϕ on $PG(r-1,q)$ by $\phi(v) = v'$ and $\phi(b_{s,t}) = d_{s,t}$, for all s and t such that $s \geqslant h$. Then $\phi(H) = H'$ and, for all i, we have $\phi(\operatorname{cl}_P(B_i)) = \operatorname{cl}_P(D_i)$, so $\phi(\operatorname{cl}_P(B_{i+1}) - \operatorname{cl}_P(B_i) - H) = \phi(\operatorname{cl}_P(B_{i+1})) - \phi(\operatorname{cl}_P(B_i)) - \phi(H) = \operatorname{cl}_P(D_{i+1}) - \operatorname{cl}_P(D_i) - H'$. Thus, $PG(r-1,q)|(\operatorname{cl}_P(G_E) - H) \cong PG(r-1,q)|(\operatorname{cl}_P(G_F) - H')$. Therefore, $AG(r-1,q)|G_E \cong AG(r-1,q)|G_F$. ☐

Similar to projective targets, the previous result means that we can refer to a simple $GF(q)$-representable affine matroid M as being an affine target when all the $GF(q)$-representations of M are affine targets.

Proposition 14. *The class of affine targets is closed under induced restrictions.*

Proof. Let (G,R) be a 2-coloring of $AG(r-1,q)$. Assume that G is an affine target. Then G corresponds to a nested sequence $(F_0, F_1, \dots, F_k)$ of affine flats with G being the union of the sets $F_{i+1} - F_i$ for all even i. Take a proper flat X of $AG(r-1,q)$. As the intersection of two affine flats is an affine flat, the sequence $(X \cap F_0, X \cap F_1, \dots, X \cap F_k)$ is a nested sequence of affine flats. Assume that n is odd. As $F_n - F_{n-1} \subseteq G$, it follows that $(X \cap F_n) - (X \cap F_{n-1}) \subseteq G \cap F$. Hence, $G \cap F$ is the union of the sets $(X \cap F_{i+1}) - (X \cap F_i)$ for all even i. Therefore, $AG(r-1,q)|(G \cap X)$ is an affine target. ☐

We will use the following well-known lemmas about affine geometries quite often in this section (see, for example [3, Exercise 6.2.2]).

Lemma 15. *$AG(r-1,q)$ can be partitioned into q hyperplanes.*

Lemma 16. *Let X and Y be distinct hyperplanes of $AG(r-1,q)$. Then either $r(X \cap Y) = 0$, or $r(X \cap Y) = r-2$.*

The techniques used for handling affine targets are similar to those that we used for projective targets. The binary case will be treated separately.

Lemma 17. *Let (G,R) be a 2-coloring of $AG(r-1,2)$ with $|G| = |R|$. Then $r(G) = r(R)$.*

Proof. Since $|G| = |R|$, we have that $|G| = 2^{r-2}$. Because the hyperplanes of $AG(r-1,2)$ have exactly 2^{r-2} elements, either $AG(r-1,2)|G$ is a hyperplane, or $r(G) = r$. Since $AG(r-1,2)|G$ is a hyperplane if and only if $AG(r-1,2)|R$ is a hyperplane, the lemma follows. ☐

Lemma 18. *Let (G,R) be a 2-coloring of $AG(r-1,2)$. Assume G is an affine target and F is a flat of $AG(r-1,2)$. Then either exactly one of $G \cap F$ and $R \cap F$ is of rank $r(F)$; or $r(G \cap F) = r(R \cap F) = r(F) - 1$, and each of $G \cap F$ and $R \cap F$ is an affine flat. Moreover, if $r(G \cap F) = r(F)$ and H_1 and H_2 are disjoint hyperplanes of $AG(r-1,2)|F$, then $r(G \cap H_1) = r(F) - 1$ or $r(G \cap H_2) = r(F) - 1$.*

Proof. Assume $r(G \cap F) < r(F)$. Then there is a rank-$(r(F) - 1)$ affine flat H_G that is contained in F and contains G. As H_G is a hyperplane of $AG(r-1, 2)|F$, there is another hyperplane H_R of $AG(r-1,2)|F$ that is complementary to H_G in F. Moreover, $H_R \subseteq R \cap F$, so $r(R \cap F) \geqslant r(F) - 1$. If there is a red point z in H_G, then $r(R \cap F) = r(F)$. Otherwise, $r(R \cap F) = r(G \cap F) = r(F) - 1$, and each of $R \cap F$ and $G \cap F$ is an affine flat.

Now suppose that $r(G \cap F) = r(F)$ and that H_1 and H_2 are disjoint hyperplanes of $AG(r-1, 2)|F$ with $r(G \cap H_1) < r(F) - 1$ and $r(G \cap H_2) < r(F) - 1$. As $AG(r-1, 2)|(G \cap F)$ is an affine target of rank $r(F)$, there is a hyperplane H' of $AG(r-1, 2)|F$ that is monochromatic green. Thus $r(R \cap F) < r(F)$. Since H' must meet both of H_1 and H_2, its intersection with each such set has rank $r(F) - 3$. Since F is green, it follows that H_1 or H_2 is green. □

Lemma 19. *Let (G, R) be a 2-coloring of $AG(r-1, q)$, where $q \geqslant 3$. Assume that G is an affine target and F is a flat of $AG(r-1, q)$. Then exactly one of $G \cap F$ and $R \cap F$ has rank $r(F)$.*

Proof. Assume $r(G \cap F) < r(F)$. Then there is a rank-$(r(F) - 1)$ affine flat H_G containing $G \cap F$. Thus $F - H_G$ does not contain any green points, so $r(R \cap F) = r(F)$. □

As with 2-colorings of $E(PG(r-1, q))$, for a 2-coloring (G, R) of $E(AG(r-1, q))$, a flat F is *green* if $r(G \cap F) = r(F)$. We call F *red* if $r(R \cap F) = r(F)$. Furthermore, a flat F of $AG(r-1, 2)$ is *half-green and half-red* if $r(G \cap F) = r(R \cap F) = r(F) - 1$. In this case, $G \cap F$ and $R \cap F$ are complementary hyperplanes of $AG(r-1, 2)|F$.

The following results show how one can get an affine target from a projective target and how to construct projective targets from affine targets.

Proposition 20. *Let (G, R) be a 2-coloring of $PG(r-1, q)$. Let H be a hyperplane of $PG(r-1, q)$. Assume that G is a projective target. Then $PG(r-1, q)|(G - H)$ is an affine target.*

Proof. As G is a projective target, G corresponds to a nested sequence $(F_0, F_1, \dots, F_k)$ of projective flats, where G is equal to the union of $F_{i+1} - F_i$ for all even i. Then $F_j - H$ is an affine flat for all j. Therefore, $(F_0 - H, F_1 - H, \dots, F_k - H)$ is a nested sequence of affine flats. Let $F'_j = F_j - H$ for all j. Then $PG(r-1, q)|(G - H)$ corresponds to the nested sequence $(F'_0, F'_1, \dots, F'_k)$ of affine flats and $G - H$ is equal to the union of $F'_{i+1} - F'_i$ for all even i. □

The following result is immediate.

Proposition 21. *Let (G, R) be a 2-coloring of $AG(r-1, q)$. Assume that G is an affine target corresponding to a nested sequence $(F_0, F_1, \dots, F_k)$ of affine flats where G is equal to the union of $F_{i+1} - F_i$ for all even i. Viewing $AG(r-1, q)$ as a restriction of $PG(r-1, q)$, the sequence $(\mathrm{cl}_P(F_0), \mathrm{cl}_P(F_1), \dots, \mathrm{cl}_P(F_k))$ is a nested sequence of projective flats and, if G_P is the projective target that is the union of $\mathrm{cl}_P(F_{i+1}) - \mathrm{cl}_P(F_i)$ for all even i, and $H = E(PG(r-1, q)) - E(AG(r-1, q))$, then $PG(r-1, q)|(G_P - H) \cong AG(r-1, q)|G$.*

We call the projective target G_P that arises from the affine target G in Proposition 21 the *standard projective target arising from* G. Now consider an affine target M_1 that arises from a green-red coloring of $PG(r-1,q)\backslash H$ where H is a projective hyperplane. Let M_2 be a projective target that arises as a green-red coloring of H. We say that M_1 and M_2 are *compatible* if the green-red coloring of $PG(r-1,q)$ induced by the colorings of M_1 and M_2 is a projective target, that is, if $PG(r-1,q)|(E(M_1)\cup E(M_2))$ is a projective target. In the previous proposition, the affine target G and the projective target $G_P\cap H$ are compatible as $PG(r-1,q)|(G\cup(G_P\cap H))$ is the projective target G_P. We now consider when $PG(r-1,q)|(E(M_1)\cup E(M_2))$ is not a standard projective target. As M_1 is an affine target, it corresponds to a canonical nested sequence $(F_0,F_1,\dots,F_k)$ of affine flats. Let F_h be the first non-empty flat in this sequence. Then $\mathrm{cl}_P(F_h)$ meets the projective hyperplane H in a rank-$(r(F_h)-1)$ projective flat T. In the construction of a standard projective target, T is monochromatic. The next result shows that, apart from the standard projective target, the only way for M_1 and M_2 to be compatible is if we modify the standard projective target by replacing T with a 2-coloring of it that is a projective target.

Proposition 22. *Let (G,R) be a 2-coloring of $PG(r-1,q)$. Let H be a projective hyperplane. Assume that $PG(r-1,q)|(G-H)$ is an affine target corresponding to a canonical nested sequence $(F_0,F_1,\dots,F_k)$ of affine flats. Assume that $PG(r-1,q)|(G\cap H)$ is a projective target corresponding to a canonical nested sequence $(S_0,S_1,\dots,S_t)$ of projective flats. Then $PG(r-1,q)|(G-H)$ and $PG(r-1,q)|(G\cap H)$ are compatible if and only if, when β is the smallest h such that $r(F_h)>0$,*

- *(i) there is an m in $\{0,1,\dots,t\}$ such that $F_\beta\cup S_m$ is a projective flat, $r(S_m)=r(F_\beta)-1$, and $PG(r-1,q)|(G\cap(F_\beta\cup S_m))$ is a projective target; and*
- *(ii) for all α in $\{1,2,\dots,k-\beta\}$, the set $F_{\beta+\alpha}\cup S_{m+\alpha}$ is a projective flat, $(F_{\beta+\alpha}\cup S_{m+\alpha})-(F_{\beta+\alpha-1}\cup S_{\beta+\alpha-1})$ is monochromatic, and $t=m+k-\beta$.*

Proof. Assume that $PG(r-1,q)|(G-H)$ and $PG(r-1,q)|(G\cap H)$ are compatible. Then $PG(r-1,q)|G$ is a projective target corresponding to a canonical nested sequence $(X_0,X_1,\dots,X_s)$ of projective flats. Thus $(X_0\cap H,X_1\cap H,\dots,X_s\cap H)$ is a nested sequence of projective flats for $PG(r-1,q)|H$, and $(X_0-H,X_1-H,\dots,X_s-H)$ is a nested sequence of affine flats for $PG(r-1,q)\backslash H$. Now,

- (a) $X_1=\emptyset$ and $X_2\cap H=\emptyset$ but $X_3\cap H\neq\emptyset$; or
- (b) $X_1=\emptyset$ and $X_2\cap H\neq\emptyset$; or
- (c) $X_1\neq\emptyset$ but $X_1\cap H=\emptyset$ and $X_2\cap H\neq\emptyset$; or
- (d) $X_1\neq\emptyset$ and $X_1\cap H\neq\emptyset$.

For the projective target $PG(r-1,q)|H$, the canonical nested sequence is $(X_2\cap H,X_3\cap H,\dots,X_s\cap H)$ in case (a) and is $(X_0\cap H,X_1\cap H,\dots,X_s\cap H)$ in the other three cases.

Let γ be the smallest h such that $X_h - H$ is non-empty. Then $\mathrm{cl}_P(X_\gamma - H)$ meets H in a projective flat of rank $r(X_\gamma) - 1$. Thus $PG(r-1,q)|(G \cap X_\gamma \cap H)$ is a projective target in $X_\gamma \cap H$ that corresponds to the canonical nested sequence $(X_2 \cap H, X_3 \cap H, \dots, X_\gamma \cap H)$ in case (a) and to the canonical nested sequence $(X_0 \cap H, X_1 \cap H, \dots, X_\gamma \cap H)$ in the other three cases.

Now $(X_\gamma - X_{\gamma-1}) - H = (X_\gamma - H) - (X_{\gamma-1} - H) = (X_\gamma - H) - \emptyset$. Thus $X_\gamma - H$ is monochromatic. Therefore, the canonical nested sequence corresponding to $PG(r-1,q)|(G-H)$ is $(X_{\gamma-1} - H, X_\gamma - H, \dots, X_s - H)$ when $X_\gamma - H$ is green and is $(\emptyset, X_{\gamma-1} - H, X_\gamma - H, \dots, X_s - H)$ when $X_\gamma - H$ is red. Thus $(F_0, F_1, \dots, F_k)$ is $(X_{\gamma-1} - H, X_\gamma - H, \dots, X_s - H)$ when $X_\gamma - H$ is green and is $(\emptyset, X_{\gamma-1} - H, X_\gamma - H, \dots, X_s - H)$ when $X_\gamma - H$ is red. We see that $F_\beta = X_\gamma - H$, that $F_\beta \cup (X_\gamma \cap H)$ is a projective flat, that $r(X_\gamma \cap H) = r(F_\beta) - 1$, and that $PG(r-1,q)|(G \cap (F_\beta \cup (X_\gamma \cap H))) = PG(r-1,q)|(G \cap X_\gamma)$. Therefore, $PG(r-1,q)|(G \cap (F_\beta \cup (X_\gamma \cap H)))$ is a projective target. Thus (i) holds. Evidently $F_{\beta+\alpha} \cup (X_{\gamma+\alpha} \cap H) = X_{\gamma+\alpha}$, so $F_{\beta+\alpha} \cup (X_{\gamma+\alpha} \cap H)$ is a projective flat for all α in $\{1, 2, \dots, k-\beta\}$. Moreover, $\gamma + k - \beta = s$ and (ii) holds.

Now suppose that (i) and (ii) hold. We know that $S_m - S_{m-1}$ and F_β are monochromatic. Then $PG(r-1,q)|G$ is a projective target for which the corresponding nested sequence is $(S_0, S_1, \dots, S_{m-1}, F_\beta \cup S_m, F_{\beta+1} \cup S_{m+1}, \dots, F_k \cup S_t)$ when the colors of $S_m - S_{m-1}$ and F_β match and is $(S_0, S_1, \dots, S_m, F_\beta \cup S_m, F_{\beta+1} \cup S_{m+1}, \dots, F_k \cup S_t)$ when the colors of $S_m - S_{m-1}$ and F_β differ. We conclude that $PG(r-1,q)|(G \cap H)$ and $PG(r-1,q)|(G-H)$ are compatible. ☐

A *minimal affine-non-target* is an affine matroid that is not an affine target such that every proper induced restriction of it is an affine target. The next result is an analog of Lemma 12.

Lemma 23. *Let (G,R) be a 2-coloring of $AG(r-1,q)$. Assume G is a rank-r minimal affine-non-target. Then $r(R) = r$.*

Proof. Assume $r(R) < r$. Then R is contained in an affine hyperplane H. As G is a minimal affine-non-target, $AG(r-1,q)|(G \cap H)$ is an affine target corresponding to a nested sequence $(F_0, F_1, \dots, F_{n-1}, H)$ of affine flats. As $R \subseteq H$, there are no red points in $E(AG(r-1,q)) - H$. Then we obtain the contradiction that G is an affine target for which a corresponding sequence of nested affine flats is $(F_0, F_1, \dots, F_{n-1}, H, E(AG(r-1,q)))$ if $H - F_{n-1} \subseteq R$ and $(F_0, F_1, \dots, F_{n-1}, E(AG(r-1,q)))$ if $H - F_{n-1} \subseteq G$. ☐

Lemma 24. *Let (G,R) be a 2-coloring of $AG(r-1,2)$. Assume G is a minimal affine-non-target of rank r. Then $AG(r-1,2)$ has a red hyperplane and a green hyperplane that are disjoint.*

Proof. Assume the lemma fails. By Lemma 23, $r(R) = r$, so we have a red hyperplane X_1 and a green hyperplane Y_1. There are affine hyperplanes X_2 and Y_2 that are complementary to X_1 and Y_1, respectively. As the lemma fails, X_2 is not green and Y_2 is not red. By assumption, X_1 and Y_1 meet in a rank-$(r-2)$ flat $F_{1,1}$. For $(i,j) \neq (1,1)$, let $F_{i,j} = X_i \cap Y_j$. As $r(F_{1,1}) = r-2$, it follows that $r(F_{i,j}) = r-2$ for each i and j. Then

$\{F_{1,1}, F_{1,2}, F_{2,1}, F_{2,2}\}$ is a partition of $AG(r-1,2)$ and there are red points in each of $F_{1,2}$ and $F_{1,1}$, and there are green points in each of $F_{1,1}$ and $F_{2,1}$. Next we show the following.

24.1. There is a red point in $F_{2,1}$.

As $AG(r-1,2)|(R \cap X_2)$ is a target and $r(G \cap X_2) < r-1$, it follows, by Lemma 18, that either $r(R \cap X_2) = r-1$, or $R \cap X_2$ and $G \cap X_2$ are affine flats of rank $r-2$. In the first case, there is certainly a red point in $F_{2,1}$. Consider the second case. Assume that $F_{2,1}$ is monochromatic green. Then $F_{2,2}$ is monochromatic red. As $r(R \cap F_{1,2}) > 0$, we see that $r(R \cap Y_2) = r-1$, so Y_2 is red, a contradiction. Thus 24.1 holds.

24.2. $F_{1,2}$ is red.

Assume that $F_{1,2}$ is not red. Then $r(R \cap F_{1,2}) < r-2$. As X_1 is red and $r(G \cap F_{1,1}) > 0$, it follows that $r(G \cap F_{1,2}) < r-2$. Thus, by Lemma 18, $R \cap F_{1,2}$ and $G \cap F_{1,2}$ are affine flats of rank $r-3$. Observe that $F_{1,1}$ is not red, otherwise $r(R \cap Y_1) = r-1$, a contradiction. Moreover, $F_{1,1}$ is not a green flat, otherwise X_1 is a green hyperplane. Thus $R \cap F_{1,1}$ and $G \cap F_{1,1}$ are affine flats of rank $r-3$. Now, as $r(R \cap X_1) = r-1$ and $|R \cap X_1| = |G \cap X_1|$, it follows by Lemma 17 that $r(G \cap X_1) = r-1$, a contradiction to Lemma 18. Therefore, 24.2 holds.

As Y_2 is not red but $F_{1,2}$ is red, $F_{2,2}$ is monochromatic green. Since $r(G \cap F_{2,1}) > 0$, we obtain the contradiction that X_2 is green. $\square$

In each of the remaining results in this section, we shall consider disjoint sets $\mathbf{X}$ and $\mathbf{Y}$ of hyperplanes of $AG(r-1,q)$ where the members of $\mathbf{X}$ and $\mathbf{Y}$ partition $E(AG(r-1,q))$. With $\mathbf{X} = \{X_1, X_2, \dots, X_q\}$ and $\mathbf{Y} = \{Y_1, Y_2, \dots, Y_q\}$, let $F_{i,j} = X_i \cap Y_j$ for all i and j.

Lemma 25. *Let (G,R) be a 2-coloring of $AG(r-1,q)$, where $q \geqslant 3$. Assume that G is a minimal affine-non-target of rank r. Then $AG(r-1,q)$ has a red hyperplane and a green hyperplane that are disjoint.*

Proof. Assume the lemma fails. By Lemma 19, each proper flat of $AG(r-1,q)$ is either red or green but not both. By Lemma 23, $r(G) = r(R) = r$, so $AG(r-1,q)$ has a red hyperplane X_1 and a green hyperplane Y_1. Then there are partitions $\{X_1, X_2, \dots, X_q\}$ and $\{Y_1, Y_2, \dots, Y_q\}$ of $E(AG(r-1,q))$ into sets $\mathbf{X}$ and $\mathbf{Y}$ of hyperplanes. By assumption, all the hyperplanes in $\mathbf{X}$ are red and all the hyperplanes in $\mathbf{Y}$ are green. As $X_1 \cap Y_1 \neq \emptyset$, it follows, by Lemma 16, that $r(F_{i,j}) = r-2$ for all i and j.

As X_1 is red, at most one of $F_{1,1}, F_{1,2}, \dots, F_{1,q-1}$, and $F_{1,q}$ is green. Thus, we may assume that $F_{1,1}, F_{1,2}, \dots, F_{1,q-2}$, and $F_{1,q-1}$ are red. As $F_{1,1}$ is red and Y_1 is green, $Y_1 - F_{1,1}$ will be monochromatic green. Similarly, $Y_2 - F_{1,2}$ will be monochromatic green. This implies that $r(G \cap X_2) = r(R \cap X_2) = r-1$, a contradiction. $\square$

The following technical lemmas show a relationship between the lines and planes of $AG(r-1,q)$ and the hyperplanes in $\mathbf{X}$ and $\mathbf{Y}$. In these lemmas, when we take closures, we are doing so in the underlying affine geometry $AG(r-1,q)$.

Lemma 26. *Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be two disjoint sets each consisting of a set of hyperplanes that partition $AG(r-1,q)$. Let x and y be distinct elements of $E(AG(r-1,q))$ such* *that* $|\{x,y\} \cap F_{i,j}| \leqslant 1$ *for all* i, j *and no member of* $\boldsymbol{X}$ *or* $\boldsymbol{Y}$ *contains* $\{x,y\}$. *Then* $|\operatorname{cl}(\{x,y\}) \cap X_i| = 1$ *and* $|\operatorname{cl}(\{x,y\} \cap Y_j| = 1$ *for all* i *and* j.

Proof. Clearly $|\operatorname{cl}(\{x,y\}) \cap X_i| \leqslant 1$ for all i, otherwise X_i contains $\{x,y\}$. As $|\operatorname{cl}(\{x,y\})| = q$, we deduce that $|\operatorname{cl}(\{x,y\}) \cap X_i| = 1$ for all i. The lemma follows by symmetry. □

Lemma 27. *For* q *in* $\{2,3\}$, *let* $\boldsymbol{X}$ *and* $\boldsymbol{Y}$ *be two disjoint sets each consisting of a set of hyperplanes that partition* $AG(r-1,q)$. *Let* $\{x,y,z\}$ *be a rank-3 subset of* $E(AG(r-1,q))$ *such that* $|\{x,y,z\} \cap F_{i,j}| \leqslant 1$ *for all* i *and* j, *and there is an* X_k *in* $\boldsymbol{X}$ *such that* $|X_k \cap \{x,y,z\}| = 2$. *Then* $|\operatorname{cl}(\{x,y,z\}) \cap F_{i,j}| = 1$ *for all* i *and* j.

Proof. Note that, by Lemma 15, each $F_{i,j}$ has rank $r-2$. Let $q=2$. We may assume that $x \in F_{1,1}$, that $y \in F_{1,2}$, and that $z \in F_{2,1}$. As $r(\{x,y,z\}) = 3$, there is exactly one point, say e, in $\operatorname{cl}(\{x,y,z\}) - \{x,y,z\}$. Assume $e \notin F_{2,2}$. Then, by symmetry, we may assume that $e \in X_1$. As $\{e,x,y,z\}$ is a circuit, we deduce that $z \in X_1$, a contradiction.

Now assume that $q=3$. We may assume that $x \in F_{1,1}$ and $y \in F_{1,2}$. Suppose $z \in F_{2,3}$. Consider $\operatorname{cl}(\{x,z\})$. The third point e on this line cannot be in X_1, otherwise the circuit $\{e,x,z\}$ gives the contradiction that z is in X_1. Similarly, e cannot be in X_2, Y_1, or Y_3. Therefore, $e \in F_{3,2}$. By a similar argument, the third point on $\operatorname{cl}(\{y,z\})$ is in $F_{3,1}$. Continuing in this manner, we deduce that $|\operatorname{cl}(\{x,y,z\}) \cap X_3| = 3$. Since $|\operatorname{cl}(\{x,y,z\})| = 9$, using the same technique, we deduce that $|\operatorname{cl}(\{x,y,z\}) \cap F_{i,j}| = 1$ for all i and j. By symmetry, we may now assume that $z \in F_{2,1}$. Then the third elements on the lines $\operatorname{cl}(\{x,z\}), \operatorname{cl}(\{x,y\})$, and $\operatorname{cl}(\{y,z\})$ are in $F_{3,1}$, $F_{1,3}$, and $F_{3,3}$, respectively. Arguing as before, we again deduce that $|\operatorname{cl}(\{x,y,z\}) \cap F_{i,j}| = 1$ for all i and j. □

Lemma 28. *Let* $\boldsymbol{X}$ *and* $\boldsymbol{Y}$ *be two disjoint sets each consisting of hyperplanes that partition* $AG(r-1,2)$. *Let* $P_1 = \{w,x,y,z\}$ *be a rank-3 flat of* $AG(r-1,2)$ *such that* $w,x \in F_{1,2}$ *and* $y,z \in F_{2,1}$. *Let* $P_2 = \{e,f,y,z\}$ *be a rank-3 flat of* $AG(r-1,2)$ *such that* $e,f \in F_{2,2}$. *Then* $\operatorname{cl}(P_1 \cup P_2)$ *is a rank-4 affine flat such that* $|\operatorname{cl}(P_1 \cup P_2) \cap F_{i,j}| = 2$ *for all* i *and* j.

Proof. As $|P_1 \cap P_2| = 2$, it follows that $r(P_1 \cup P_2) = 4$. Thus $\operatorname{cl}(P_1 \cup P_2)$ is a rank-4 affine flat. Now consider $\operatorname{cl}(\{e,w,z\})$. By Lemma 27, $\operatorname{cl}(\{e,w,z\})$ intersects $F_{1,1}$ in an affine flat. Therefore, as $\operatorname{cl}(P_1 \cup P_2)$ meets each of $X_1, X_2, Y_1, Y_2, F_{1,1}, F_{1,2}, F_{2,1}$, and $F_{2,2}$ in an affine flat, each such intersection has 1, 2, or 4 elements. Thus the lemma follows. □

We now prove the main results of this section.

Proof of Theorem 3. Assume G is an affine target and there is a rank-4 affine flat F such that $AG(r-1,2)|(G \cap F) \cong U_{4,4}$. Then $AG(r-1,2)|(R \cap F) \cong U_{4,4}$. This contradicts Lemma 18 as $r(G \cap F) = r(R \cap F) = 4$. Hence a binary affine target does not have $U_{4,4}$ as an induced restriction.

Let (G,R) be a 2-coloring of $AG(r-1,2)$. Suppose that G is a rank-r minimal affine-non-target and that G does not contain $U_{4,4}$ as an induced restriction. By Lemma 24, $AG(r-1,2)$ has a red hyperplane X_1 and a green hyperplane X_2 such that $X_1 \cap X_2 = \emptyset$. By Lemma 23, $r(R) = r$, so there is a red point z in X_2. As $AG(r-1,2)|(R \cap X_1)$ is an affine target, in X_1, there is a monochromatic red rank-$(r-2)$ flat $F_{1,1} \subseteq X_1$. Observe that

$\mathrm{cl}(F_{1,1} \cup z)$ is a red hyperplane Y_1 that intersects X_1 and X_2. Then there is a hyperplane Y_2 that is complementary to Y_1. By Lemma 16, $r(F_{i,j}) = r - 2$ for all i and j. Observe that z is in $F_{2,1}$. Furthermore, there is a red point e in $F_{1,2}$ and there are green points f and g in $F_{2,1}$ and $F_{2,2}$, respectively. As $r(G) = r$, there is a green point h in $F_{1,2}$. We make the following observations.

3.1. *$F_{1,2} \cup F_{2,1}$ is an affine hyperplane.*

Observe that $F_{2,1}$ is contained in three affine hyperplanes, two of which are $F_{1,1} \cup F_{2,1}$ and $F_{2,1} \cup F_{2,2}$. Therefore, $F_{1,2} \cup F_{2,1}$ is the third such hyperplane. Thus 3.1 holds.

3.2. *$F_{2,2}$ is not monochromatic green.*

Assume that $F_{2,2}$ is monochromatic green. Then Y_2 is green. By 3.1, $F_{1,2} \cup F_{2,1}$ is an affine hyperplane, so both $AG(r-1,2)|(G \cap (F_{1,2} \cup F_{2,1}))$ and $AG(r-1,2)|(R \cap (F_{1,2} \cup F_{2,1}))$ are affine targets. Then there is a rank-$(r-2)$ affine flat F such that either $G \cap (F_{1,2} \cup F_{2,1}) \subseteq F$ or $R \cap (F_{1,2} \cup F_{2,1}) \subseteq F$. Because we currently have symmetry between the red and green subsets of $AG(r-1,2)$, we may assume the former. Then f and h are in F. Let x be a red point in $F_{1,2} - F$. Let $P_1 = \mathrm{cl}(\{f,h,x\})$. The fourth point y on this plane is in Y_1, otherwise the circuit $\{f,h,x,y\}$ gives the contradiction that $f \in Y_2$. Moreover, $y \notin F$, otherwise the circuit $\{f,h,x,y\}$ gives the contradiction that $x \in F$. Thus $y \in F_{2,1} - F$, so y is red. Let $P_2 = \mathrm{cl}(\{f,g,y\})$. Then the fourth point g' on this plane is in $F_{2,2}$, so g' is green. By Lemma 28, $r(\mathrm{cl}(P_1 \cup P_2)) = 4$. Let $\{s,t\} = \mathrm{cl}(P_1 \cup P_2) - \{f,g,g',h,x,y\}$. Then, by Lemma 28, s and t in $F_{1,1}$, so both points are red. Therefore, $r(G \cap \mathrm{cl}(P_1 \cup P_2)) = r(R \cap \mathrm{cl}(P_1 \cup P_2)) = 4$, so $AG(r-1,2)|\{f,g,g',h\} \cong U_{4,4}$. We conclude that $AG(r-1,2)|G$ has $U_{4,4}$ as an induced restriction, a contradiction. Thus 3.2 holds.

The affine hyperplane $F_{1,2} \cup F_{2,1}$ is either green, red, or half-green and half-red.

3.3. *$F_{1,2} \cup F_{2,1}$ is not red*

Assume that $F_{1,2} \cup F_{2,1}$ is red. Then, by the last part of Lemma 18, at least one of $F_{1,2}$ and $F_{2,1}$ will be red. Assume that $F_{2,1}$ is red. As X_2 is green, $F_{2,2}$ is monochromatic green, otherwise $r(R \cap X_2) = r - 1$. By 3.2, we deduce that $F_{2,1}$ is not red. Thus $F_{1,2}$ is red. Observe that if $F_{2,1}$ is green, then $r(G \cap (F_{1,2} \cup F_{2,1})) = r(R \cap (F_{1,2} \cup F_{2,1})) = r - 1$, a contradiction. Thus, $F_{2,1}$ is half-green and half-red. As X_2 is green, the last part of Lemma 18 gives that $F_{2,2}$ is green. Therefore $r(G \cap (F_{2,2} \cup h)) = r - 1$, so Y_2 is green. As $F_{1,2}$ is red, $F_{2,2}$ is monochromatic green, a contradiction to 3.2. Therefore, 3.3 holds.

Since $G \cap (F_{1,2} \cup F_{2,1})$ is an affine target and $F_{1,2} \cup F_{2,1}$ is not red, there is a monochromatic green flat Z of rank $r-2$ that is contained in $F_{1,2} \cup F_{2,1}$. Because neither $F_{1,2}$ nor $F_{2,1}$ is monochromatic green, Z meets $F_{1,2}$ and $F_{2,1}$ in monochromatic green flats, $Z_{1,2}$ and $Z_{2,1}$, of rank $r-3$. Similarly, as $G \cap X_2$ is an affine target and X_2 is green, there is a monochromatic green flat V of rank $r-2$ that is contained in X_2. Because neither $F_{2,1}$ nor $F_{2,2}$ is monochromatic green, V meets $F_{2,1}$ and $F_{2,2}$ in monochromatic green flats, $V_{2,1}$ and $V_{2,2}$, of rank $r-3$.

In the next part of the argument, we shall use the observation that if Y_2 is green, then we have symmetry between (X_1, X_2) and (Y_1, Y_2).

3.4. *$F_{1,2} \cup F_{2,1}$ is not green.*

Assume that $F_{1,2} \cup F_{2,1}$ is green. Then, by Lemma 18, $F_{2,1}$ or $F_{1,2}$ is green. But the latter case implies that Y_2 is green, so this case can be reduced to the former by the symmetry between (X_1, X_2) and (Y_1, Y_2) noted above. Thus we may assume that $F_{2,1}$ is green.

Now $Z_{2,1}$ and $V_{2,1}$ are rank-$(r-3)$ monochromatic green flats that are both contained in $F_{2,1}$. Suppose $Z_{2,1} = V_{2,1}$. As $F_{2,1}$ is green, there is a green element g_1 in $F_{2,1} - Z_{2,1}$. Since $F_{2,2}$ is not monochromatic green, there is a red point u_1 in $F_{2,2} - V_{2,2}$. Take a green point g_2 in $V_{2,2}$ and let $P_1 = \mathrm{cl}(\{g_1, g_2, u_1\})$. Let the fourth point on this plane be g_3. Then $g_3 \in F_{2,1}$, otherwise the circuit $\{g_1, g_2, g_3, u_1\}$ implies that $g_1 \in F_{2,2}$, a contradiction. Likewise, $g_3 \in V_{2,1}$, otherwise $g_2 \notin V$, a contradiction. Because X_1 is red, there is a red point u_2 in $F_{1,2} - Z_{1,2}$. Let $P_2 = \mathrm{cl}(\{g_1, g_3, u_2\})$. Let g_4 be the fourth point on this plane. Then the circuit $\{g_1, g_3, g_3, u_2\}$ implies that $g_4 \in Z_{1,2}$, so g_4 is green. By Lemma 28, $\mathrm{cl}(P_1 \cup P_2)$ is a rank-4 affine flat having two points, s and t, in $F_{1,1}$. We see that $AG(r-1, 2)|\{g_1, g_2, g_3, g_4\} \cong U_{4,4}$. Thus G has $U_{4,4}$ as an induced restriction, a contradiction. Thus $Z_{2,1} \neq V_{2,1}$.

Now Y_2 contains the monochromatic green flats $Z_{1,2}$ and $V_{2,2}$, each of which has rank $r-3$. Thus Y_2 is green, or Y_2 is half-green and half-red. Assume the latter. Then $Z_{1,2} \cup V_{2,2}$ is a monochromatic green flat of rank $r-2$ and $Y_2 - (Z_{1,2} \cup V_{2,2})$ is a monochromatic red flat of rank $r-2$. As before, we take u_1 to be a red point in $F_{2,2}$. Choose g_1 to be a point in $V_{2,2}$. Then g_1 is green. Let g_2 be a point in $Z_{2,1} - V_{2,1}$, so g_2 is green. Let $P_1 = \mathrm{cl}(\{g_1, g_2, u_1\})$ and let g_3 be the fourth point in P_1. Then $g_3 \in F_{2,1}$ and $g_3 \in V$. Thus $g_3 \in V_{2,1}$, so g_3 is green. Choose u_2 in $F_{1,2} - Z_{1,2}$. Then u_2 is red. Let $P_2 = \mathrm{cl}(\{g_1, u_1, u_2\})$ and let g_4 be the fourth point in P_2. Then $g_4 \in F_{1,2}$ and $g_4 \in Z_{1,2} \cup V_{2,2}$, so $g_4 \in Z_{1,2}$. Thus g_4 is green. By Lemma 28, $\mathrm{cl}(P_1 \cup P_2)$ is a rank-4 affine flat that meets $F_{1,1}$ in two elements, both of which are red. Moreover, $AG(r-1, 2)|\{g_1, g_2, g_3, g_4\} \cong U_{4,4}$, a contradiction.

We now know that Y_2 is green. Then there is a monochromatic green flat W of rank $r-2$ such that $W \subseteq Y_2$. As neither $F_{1,2}$ nor $F_{2,2}$ is monochromatic green, $W \cap F_{1,2}$ and $W \cap F_{2,2}$ are monochromatic green flats, $W_{1,2}$ and $W_{2,2}$, of rank $r-3$. We choose u_1 to be a red point in $F_{2,2} - (V_{2,2} \cup W_{2,2})$. Choose g_1 in $V_{2,2} \cap W_{2,2}$. Then g_1 is green. Choose g_2 in $Z_{2,1} - V_{2,1}$. Then g_2 is green. The fourth point g_3 of the plane P_1 that equals $\mathrm{cl}(\{g_1, g_2, u_1\})$ is in $F_{2,1} \cap V$; that is, $g_3 \in V_{2,1}$, so g_3 is green. Now let u_2 be a red point in $F_{1,2} - (Z_{1,2} \cup W_{1,2})$. The fourth point g_4 on the plane P_2 that equals $\mathrm{cl}(\{g_1, u_1, u_2\})$ is in $F_{1,2} \cap W$, so it is in $W_{1,2}$ and hence is green. Then, by Lemma 28, $\mathrm{cl}(P_1 \cup P_2)$ is a rank-4 affine flat that contains exactly four green points g_1, g_2, g_3, and g_4. Since $AG(r-1, 2)|\{g_1, g_2, g_3, g_4\} \cong U_{4,4}$, we have a contradiction. We conclude that 3.4 holds.

By 3.3 and 3.4, we must have that $F_{1,2} \cup F_{2,1}$ is half-green and half-red. As Z is a monochromatic green flat of rank $r-2$ that is contained in $F_{1,2} \cup F_{2,1}$, we deduce that $(F_{1,2} \cup F_{2,1}) - Z$ is a monochromatic red flat of rank $r-2$. Moreover, $F_{1,2} - Z$ and $F_{2,1} - Z$ are monochromatic red flats of rank $r-3$. Thus $V_{2,1} = Z_{2,1}$. As X_2 is green, there is a green point g_1 in $F_{2,2} - V$. Take g_2 to be a point in $V_{2,2}$ and let u_1 be a point in $F_{2,1} - V_{2,1}$. Let $P_1 = \mathrm{cl}(\{g_1, g_2, u_1\})$. The fourth point g_3 on this plane is in $F_{2,2}$ and in V so it is in $V_{2,2}$ and hence it is green. Let u_2 be a point in $F_{1,2} - Z_{1,2}$. Then u_2 is red. Let $P_2 = \mathrm{cl}(\{g_3, u_1, u_2\})$. The fourth point g_4 on this plane is in $F_{1,2} \cap Z$, so it is green. By Lemma 28, $\mathrm{cl}(P_1 \cup P_2)$ is a rank-4 affine flat that contains exactly four green points, g_1, g_2, g_3, and g_4. Moreover, $AG(r-1,2)|\{g_1, g_2, g_3, g_4\} \cong U_{4,4}$, a contradiction. We conclude that the theorem holds.

□

Proof of Theorem 4. Assume that G is an affine target over $GF(3)$ such that there is a rank-3 affine flat F for which $AG(r-1,3)|(G \cap F)$ is one of $U_{3,3}, U_{3,4}, U_{2,3} \oplus U_{1,1}, U_{2,3} \oplus_2 U_{2,4}, P(U_{2,3}, U_{2,3})$, or $\mathcal{W}^3$. Then $r(G \cap F) = r(R \cap F) = 3$, contradicting Lemma 19.

Let (G, R) be a 2-coloring of $AG(r-1,3)$. Suppose that G is a rank-r minimal affine-non-target. Then $r(G) \geqslant 3$. If $r(G) = 3$, then, by Lemma 23, $r(R) = 3$. One can now check that $AG(r-1,3)|G$ is one of $U_{3,3}, U_{3,4}, U_{2,3} \oplus U_{1,1}, U_{2,3} \oplus_2 U_{2,4}, P(U_{2,3}, U_{2,3})$, or $\mathcal{W}^3$. Thus we may assume $r(G) \geqslant 4$ and that G does not contain a rank-3 flat F such that $r(G \cap F) = r(R \cap F) = 3$. By Lemma 23, $r(R) = r$. Now, by Lemma 25, there is a green hyperplane X_1 and a red hyperplane X_2 that are disjoint. Let $\{X_1, X_2, X_3\}$ and $\{Y_1, Y_2, Y_3\}$ be distinct sets, $\mathbf{X}$ and $\mathbf{Y}$, each consisting of three disjoint hyperplanes in $AG(r-1,3)$. Then, by Lemma 16, $r(F_{i,j}) = r-2$ for all i and j. We proceed by showing there are no possible colorings of the hyperplanes in $\mathbf{Y}$.

4.1. If $F_{1,1}$ and $F_{1,2}$ are green, then Y_1 or Y_2 is green.

Assume that Y_1 and Y_2 are both red. Then $Y_1 - F_{1,1}$ and $Y_2 - F_{1,2}$ are monochromatic red. As $r(G) = r$, there is a green element e in $Y_3 - F_{1,3}$. Let f and g be green elements in $F_{1,1}$ and $F_{1,2}$, respectively. Consider $\mathrm{cl}(\{e, f, g\})$. By Lemma 27 this plane will contain red points in $F_{2,1}, F_{2,2}$, and $F_{3,2}$. Therefore $r(G \cap \mathrm{cl}(\{e, f, g\})) = r(R \cap \mathrm{cl}(\{e, f, g\})) = 3$, a contradiction. Thus 4.1 holds.

4.2. There cannot be at least two red hyperplanes or at least two green hyperplanes in $\mathbf{Y}$.

Assume that Y_1 and Y_2 are red. As X_1 is green, at most one of $F_{1,1}, F_{1,2}$, and $F_{1,3}$ is red. By 4.1, we may assume that $F_{1,2}$ is red. Then $X_1 - F_{1,2}$ is monochromatic green, so $Y_1 - F_{1,1}$ is monochromatic red. As $r(G) = r$, there is a green point g that is not in X_1. Then $g \in F_{2,2} \cup F_{2,3} \cup F_{3,2} \cup F_{3,3}$. Let e be a green point in $F_{1,1}$, and f be a red point in $F_{1,2}$. Consider $\mathrm{cl}(\{e, f, g\})$. By Lemma 27, this plane will contain red points in $F_{2,1}$ and $F_{3,1}$, and a green point in $F_{1,3}$. Therefore, $r(G \cap \mathrm{cl}(\{e, f, g\})) = r(R \cap \mathrm{cl}(\{e, f, g\})) = 3$, a contradiction. By symmetry, there cannot be two green hyperplanes in $\mathbf{Y}$. Thus 4.2 holds.

We conclude that there are no possible colorings of the hyperplanes in $\mathbf{Y}$, a contradiction. □

Proof of Theorem 5. Assume that G is an affine target over $GF(q)$ for $q \geqslant 4$. If there is a rank-2 affine flat F such that $AG(r-1,q)|(G \cap F)$ is any of $U_{2,2}, U_{2,3}, \ldots, U_{2,q-3}$, or $U_{2,q-2}$, then $r(G \cap F) = r(R \cap F)$, contradicting Lemma 19.

Let (G, R) be a 2-coloring of $AG(r-1,q)$. Suppose that G is a rank-r minimal affine-non-target that does not contain $U_{2,2}, U_{2,3}, \ldots, U_{2,q-3}$, or $U_{2,q-2}$ as an induced restriction. Then $r(G) \geqslant 3$. By Lemma 19, $r(R) = r$. Now, by Lemma 25, there is a red hyperplane

X_1 that is disjoint from a green hyperplane X_2. Let $\{X_1, X_2, \ldots, X_q\}$ and $\{Y_1, Y_2, \ldots, Y_q\}$ be disjoint sets, $\mathbf{X}$ and $\mathbf{Y}$, each consisting of q disjoint hyperplanes in $AG(r-1, q)$. By Lemma 16, $r(F_{i,j}) = r-2$ for all i and j. We show there are no possible colorings of the hyperplanes in $\mathbf{Y}$.

5.1. There is at least one green hyperplane and at least one red hyperplane in $\mathbf{Y}$.

Assume that all members of $\mathbf{Y}$ are red. As X_2 is green, we may assume that $F_{2,1}, F_{2,2}, \ldots, F_{2,q-2}$, and $F_{2,q-1}$ are green. Then $Y_k - F_{2,k}$ is monochromatic red for all k in $\{1, 2, \ldots, q-1\}$. As $r(G) = r$, there is a green element e in $Y_q - F_{2,q}$. We may assume that $e \in F_{3,q}$. Let f be a green element in $F_{2,1}$. Consider $\mathrm{cl}(\{e, f\})$. By Lemma 26, this line will contain red points in $Y_2 - (F_{2,2} \cup F_{3,2})$ and $Y_3 - (F_{2,3} \cup F_{3,3})$. However, this gives the contradiction that $r(G \cap \mathrm{cl}(\{e, f\})) = r(R \cap \mathrm{cl}(\{e, f\})) = 2$. By symmetry, not all members of $\mathbf{Y}$ are green. Thus 5.1 holds.

5.2. There cannot be at least two green hyperplanes and at least two red hyperplanes in $\mathbf{Y}$.

Let Y_1 and Y_2 be green and let Y_3 and Y_4 be red. As X_1 is red, at most one of $F_{1,1}, F_{1,2}, \ldots, F_{1,q-1}$, and $F_{1,q}$ is green. This implies that $F_{1,1}$ or $F_{1,2}$ is red, so we may assume the latter. Then $Y_2 - F_{1,2}$ is monochromatic green. Similarly, as X_2 is green, $F_{2,3}$ or $F_{2,4}$, say $F_{2,3}$, is green. Then $Y_3 - F_{2,3}$ is monochromatic red. Assume $F_{1,1}$ and $F_{2,1}$ are green. Then $X_1 - F_{1,1}$ is monochromatic red. Let e be a red point in $F_{1,4}$ and let f be a green point in $F_{2,1}$. Consider $\mathrm{cl}(\{e, f\})$. Then, by Lemma 26, this line will have a green point in $Y_2 - (F_{1,2} \cup F_{2,2})$ and a red point in $Y_3 - (F_{1,3} \cup F_{2,3})$. Hence $r(G \cap \mathrm{cl}(\{e, f\})) = r(R \cap \mathrm{cl}(\{e, f\}))$, a contradiction. A symmetric argument holds when $F_{1,4}$ and $F_{2,4}$ are both red. Therefore, $F_{1,1}$ or $F_{2,1}$ is red, and $F_{1,4}$ or $F_{2,4}$ is green. This implies that $Y_1 - (F_{1,1} \cup F_{2,1})$ is monochromatic green and $Y_4 - (F_{1,4} \cup F_{2,4})$ is monochromatic red. Hence $r(G \cap X_3) = r(R \cap X_3) = r-1$, a contradiction. Thus 5.2 holds.

5.3. There cannot be exactly one red hyperplane or exactly one green hyperplane in $\mathbf{Y}$.

Assume that Y_1 is red and $Y_2, Y_3, \ldots, Y_{q-1}$, and Y_q are green. As X_1 is red, at most one of $F_{1,1}, F_{1,2}, \ldots, F_{1,q-1}$, and $F_{1,q}$ is green. First assume that $F_{1,2}, F_{1,3}, \ldots, F_{1,q-1}$ and $F_{1,q}$ are red. Then $Y_k - F_{1,k}$ is monochromatic green for all k in $\{2, 3, \ldots, q\}$. By Lemma 19, $r(R) = r$, so there is a red point e in $Y_1 - F_{1,1}$. We may assume e is in $F_{2,1}$. Let f be a red point in $F_{1,2}$ and consider $\mathrm{cl}(\{e, f\})$. By Lemma 26, this line will have green points in $X_3 - (F_{3,1} \cup F_{3,2})$ and $X_4 - (F_{4,1} \cup F_{4,2})$. Therefore, $r(G \cap \mathrm{cl}(\{e, f\})) = r(R \cap \mathrm{cl}(\{e, f\})) = 2$, a contradiction.

Now assume that $F_{1,2}$ is green. Then $X_1 - F_{1,2}$ is monochromatic red. Hence $Y_k - F_{1,k}$ is monochromatic green for all k in $\{3, 4, \ldots, q\}$. Let e be a red element in $Y_1 - F_{1,1}$ and f be a green element in $Y_2 - F_{1,2}$ such that $|X_i \cap \{e, f\}| \leqslant 1$ for all i in $\{2, 3, \ldots, q\}$. As Y_1 is red and Y_2 is green, such a pair of points exists. We may assume that $e \in F_{2,1}$ and $f \in F_{3,2}$. Then, by Lemma 26, $\mathrm{cl}(\{e, f\})$ will contain a red element in $X_1 - (F_{1,1} \cup F_{1,2})$ and a green element in $X_4 - (F_{4,1} \cup F_{4,2})$, a contradiction. By symmetry, there cannot be exactly one green hyperplane in $\mathbf{Y}$. Thus 5.3 holds.

We conclude that there are no possible colorings of the hyperplanes in $\mathbf{Y}$, a contradiction. □

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