# Remarks on Tracking and Robustness Analysis for MEM Relays

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Joint with Frédéric Mazenc and Marcio de Queiroz

2008 American Control Conference

# OUTLINE

- Types of Relays
- System Model and Reference Trajectories
- Error Dynamics and Objective
- Stability Theorem
- Crafting the Reference Trajectory
- Numerical Validation
- Conclusions

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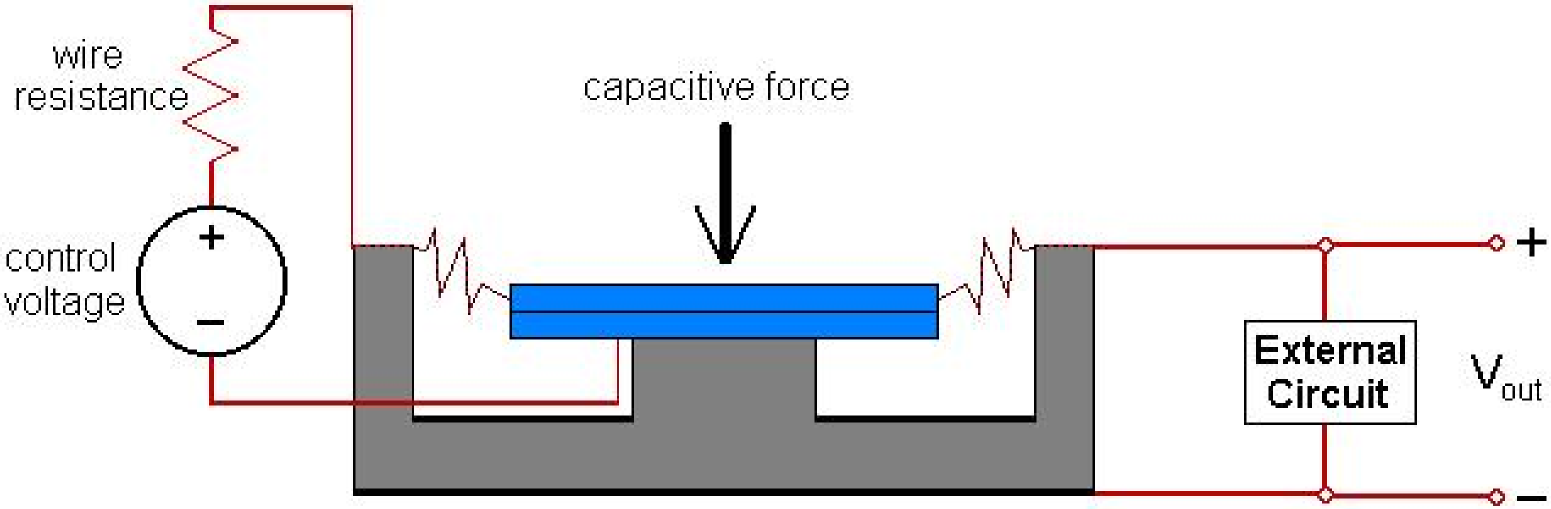
# ELECTROSTATIC and ELECTROMAGNETIC MEM RELAYS

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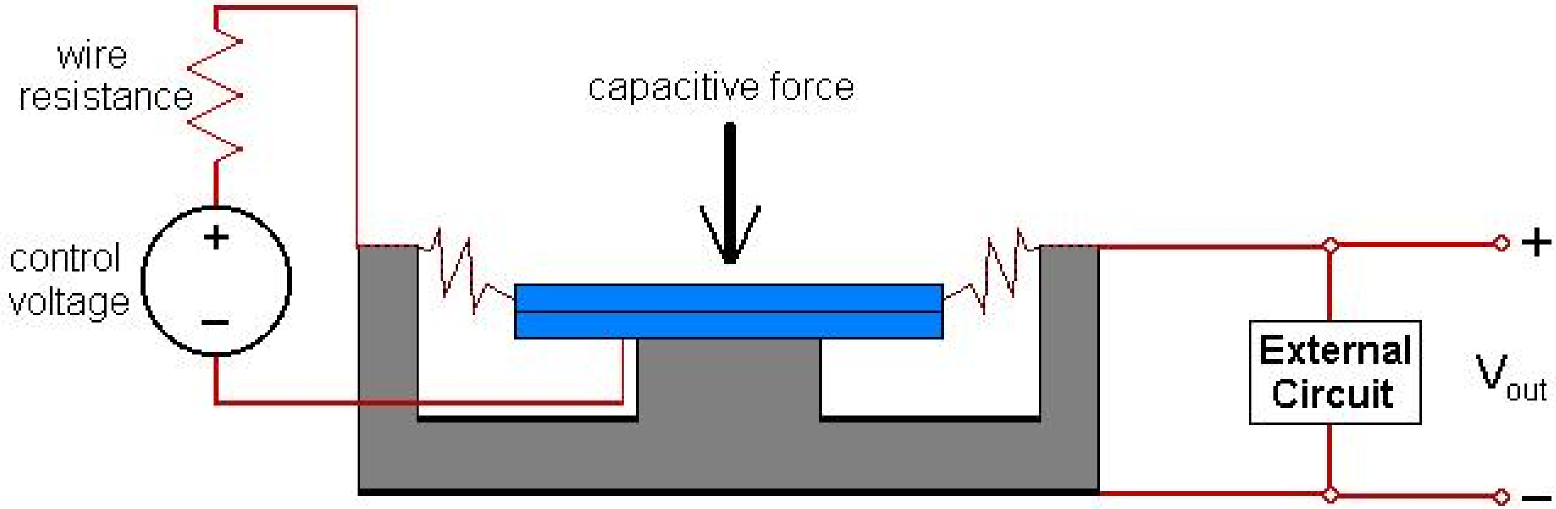


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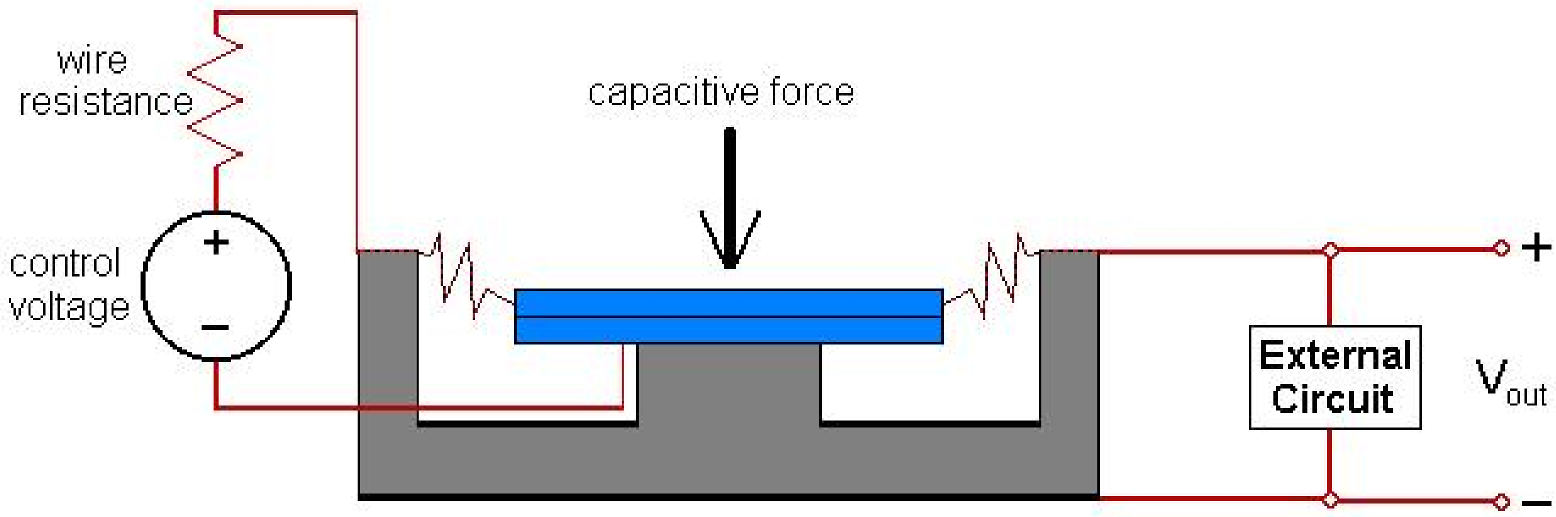


- Control and output circuits use a common pair of parallel electrodes, one movable and one fixed.
- Voltage applied across electrodes creates an attractive capacitive or magnetic force between them.

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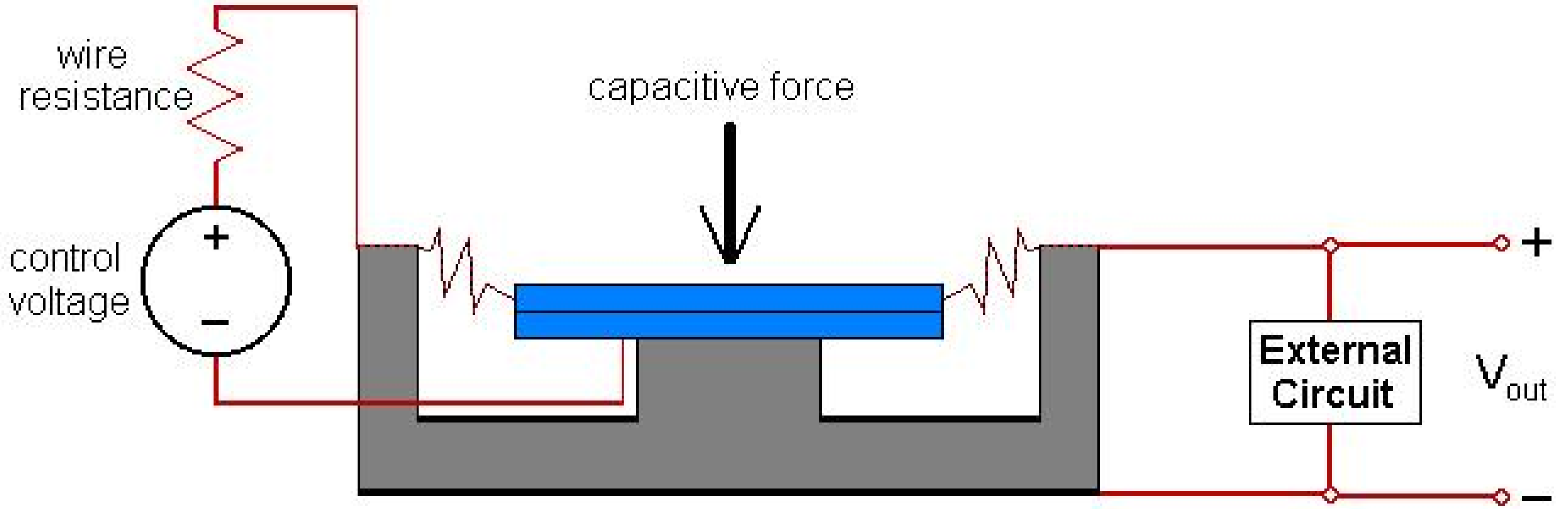


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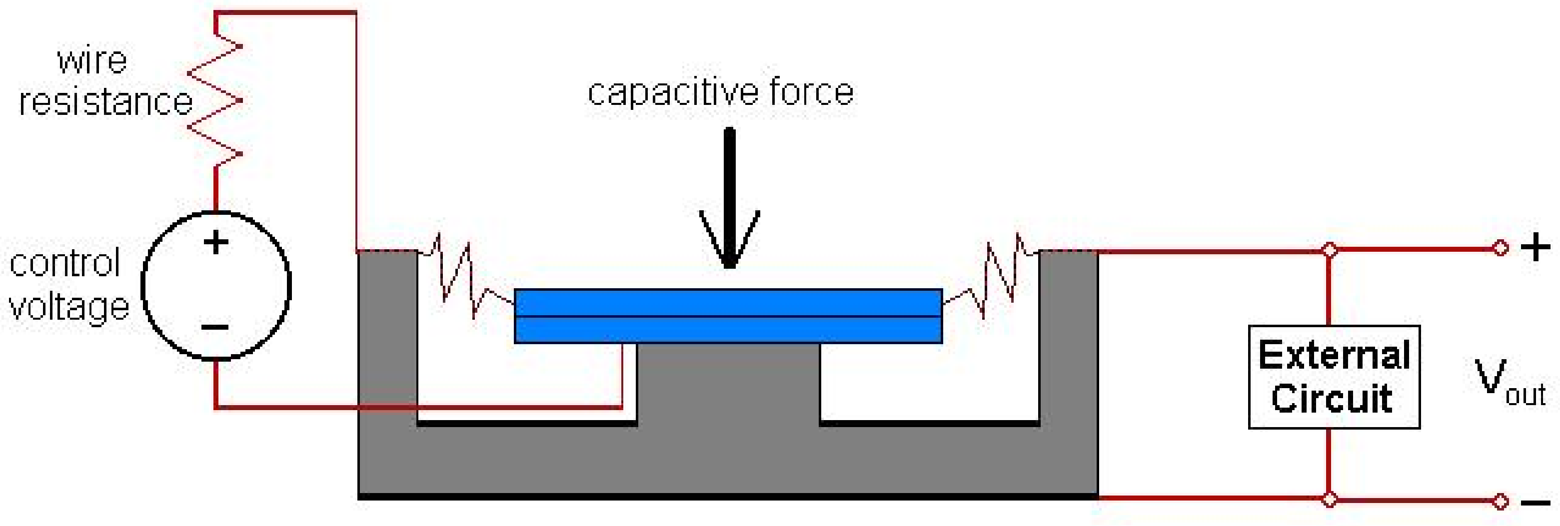


- Electrodes come together as do the contacts of the output circuit.
- In both types, pull-in occurs, i.e., the movable electrode suddenly ‘crashes’ onto the bottom electrode, so feedback control is necessary.

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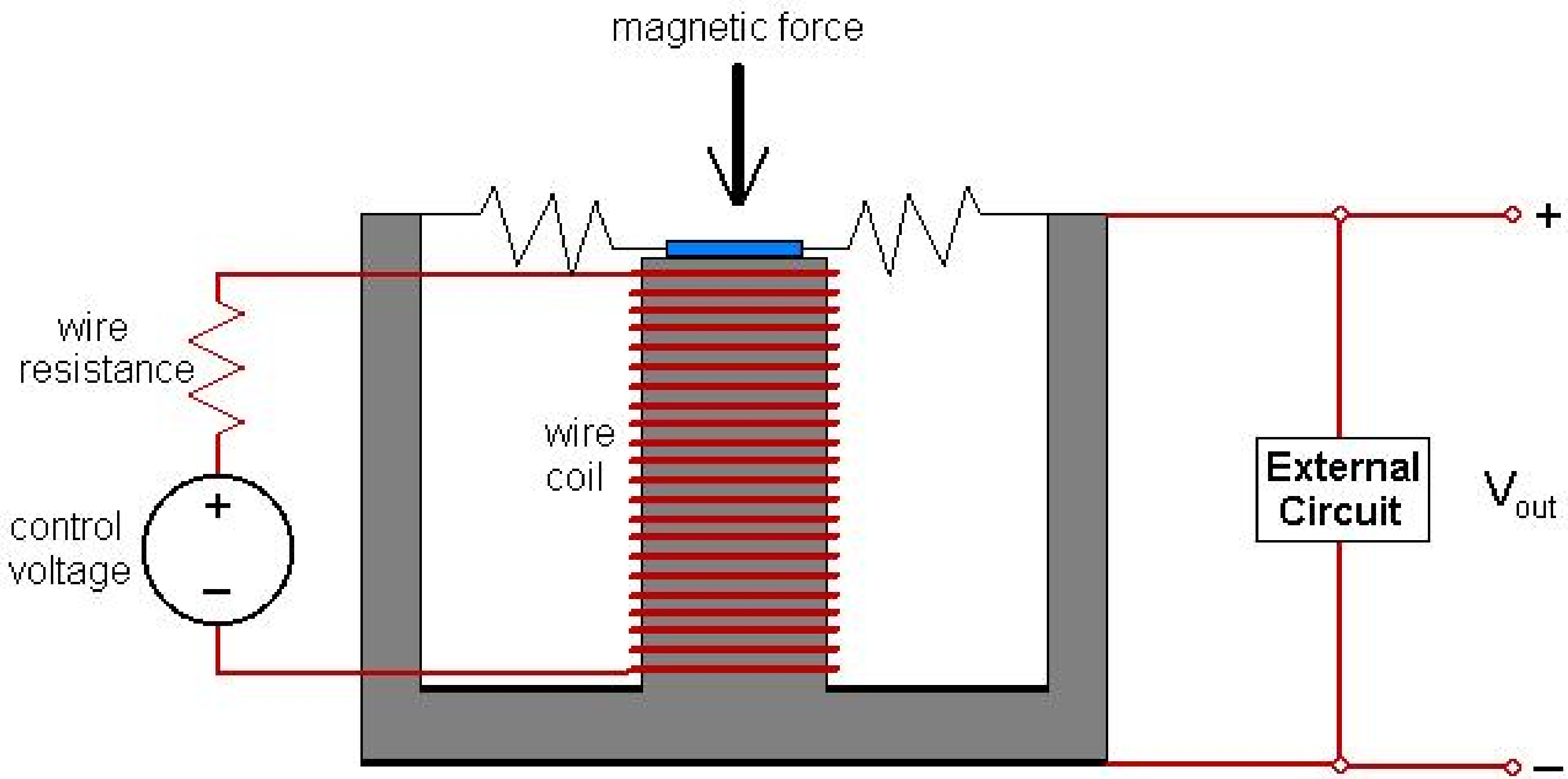

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- Zhu-Praly-.. ’05: differential flatness, backstepping.
- Younis-Gao-de Queiroz ’07: Lyapunov-based setpoint controller, feedback linearization tracking controller.

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## SYSTEM MODEL and REFERENCE TRAJECTORIES

Unifying Model (Younis-Gao-de Queiroz, ACC 07):

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\begin{aligned}
m\ddot{x} + b\dot{x} + kx &= \alpha z^2 \\
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upper (resp., lower) corresponds to electrostatic (resp., electromagnetic)

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$m =$ movable electrode mass

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$m =$ movable electrode mass, $b =$ squeezed-film damping

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$k =$ spring stiffness,

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$A$ = movable electrode area, $R$ = resistance of circuit

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Electrostatic: $q =$ charge

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Electromagnetic: $\phi =$ flux, $\mu =$ gap permeability

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Electrostatic: $q =$ charge, $\epsilon =$ gap permittivity.

Electromagnetic: $N =$ # coil turns.

# TRACKING

Unifying Model (Younis-Gao-de Queiroz, ACC 07):

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Reference Trajectory: $C^3 \ni y_d : [0, \infty) \to \mathbb{R}$ s.t. $\exists$ constants $m_i > 0$ for which

(a) $m_1 \leq y_d(t) \leq m_2$, $|\dot{y}_d(t)| \leq m_3$, and $|\ddot{y}_d(t)| \leq m_4 \ \forall t$

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and (b) $m_4 + \frac{b}{m} m_3 < 0.9 \frac{k}{m} m_1$.

# GOAL

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Goal: Design a controller $u(x, \dot{x}, z, t)$ that forces $x(t)$ to track $y_d(t)$ for all initial states $x(t_0) \in (-\infty, g_o)$.

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Reference trajectory, instead of a set point, can improve micro-relay performance.

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## ERROR DYNAMICS

Setting $e_1 = x - y_d$ and $e_2 = \dot{e}_1$ and using the change of feedback $u = \gamma\,(g_0 - x)\,z + \beta v_1 \sqrt{m/\alpha}$ gives

$$\left\{\begin{array}{rcl} \dot{e}_1 & = & e_2 \\ \dot{e}_2 & = & -\kappa_1 e_1 - \kappa_2 e_2 + \mu(e_1, e_2) + \zeta^2 + 2\zeta R_\mu(e_1, e_2, t) \\ \dot{\zeta} & = & v_1 - \frac{1}{2R_\mu(e_1,e_2,t)}\left\{\dddot{y}_d(t) + \kappa_2 \ddot{y}_d(t) + \kappa_1 \dot{y}_d(t) + \dot{\mu}\right\} \end{array}\right.$$

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$$|\mu(e_1, e_2)| \leq 0.1 \kappa_1 m_1$$

# ERROR DYNAMICS

Setting $e_1 = x - y_d$ and $e_2 = \dot{e}_1$ and using the change of feedback $u = \gamma\,(g_0 - x)\,z + \beta v_1 \sqrt{m/\alpha}$ gives

$$\left\{ \begin{array}{lcl} \dot{e}_1 & = & e_2 \\ \dot{e}_2 & = & -\kappa_1 e_1 - \kappa_2 e_2 + \mu(e_1, e_2) + \zeta^2 + 2\zeta R_\mu(e_1, e_2, t) \\ \dot{\zeta} & = & v_1 - \frac{1}{2R_\mu(e_1,e_2,t)} \{ \dddot{y}_d(t) + \kappa_2 \ddot{y}_d(t) + \kappa_1 \dot{y}_d(t) + \dot{\mu} \} \end{array} \right.$$

where

$$\zeta(t) = \sqrt{\tfrac{\alpha}{m}} z(t) - R_\mu(e_1(t), e_2(t), t), \quad \kappa_1 = \tfrac{k}{m}, \quad \kappa_2 = \tfrac{b}{m},$$

$$|\mu(e_1, e_2)| \leq 0.1 \kappa_1 m_1, \quad \text{and}$$

$$R_\mu(e_1, e_2, t) = \sqrt{\ddot{y}_d(t) + \kappa_2 \dot{y}_d(t) + \kappa_1 y_d(t) + \mu(e_1, e_2)}.$$

## DESIRED STABILITY

Goal: For each constant $\mathcal{L} > 0$, design $\mu \in C^1$ and $v_1$ for which:

G1 The closed loop $Y = (e_1, e_2, \zeta)$ system is UGAS to $0$.

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G2 There exist constant $\underline{K}, \overline{K} > 0$ such that for all closed loop solutions with initial states $Y(t_o) \in \underline{K}\mathcal{B}_3$, we have

$$|Y(t)| \ \leq \ \overline{K} e^{-\mathcal{L}(t-t_o)} |Y(t_o)| \quad \forall t \geq t_o \geq 0.$$

# OUTLINE

- Types of Relays
- System Model and Reference Trajectories
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# STABILITY THEOREM

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Then we can compute a constant $\underline{a}(a_1, a_2)$ such that for each $a_3 \geq \underline{a}$,

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renders the error dynamics UGAS to the origin. Moreover, for each constant $\mathcal{L} > 0$, we can choose values of the $a_i$s and $\underline{K}, \overline{K} > 0$ to satisfy G2.

# IDEA of PROOF

Construct explicit constants $K$ and $\Gamma$ so that the $Y = (e_1, e_2, \zeta)$ dynamics has the strict global Lyapunov function

$$V_3(e_1, e_2, \zeta) = V_2(e_1, e_2) + \Gamma Q(\zeta), \quad \text{where}$$

$$V_2(e_1, e_2) = e_1 e_2 + K V_1(e_1, e_2),$$

$$Q(\zeta) = \frac{1}{a_3}\left(\frac{1}{2}\zeta^2 + \frac{1}{4}\zeta^4\right),$$

$$\text{and} \;\; V_1(e_1, e_2) = \frac{1}{2}e_2^2 + \int_0^{e_1} \left\{ \kappa_1 l + \frac{\kappa_1 m_1}{20} \sigma\left(\frac{20 a_1}{\kappa_1 m_1} l\right) \right\} dl.$$

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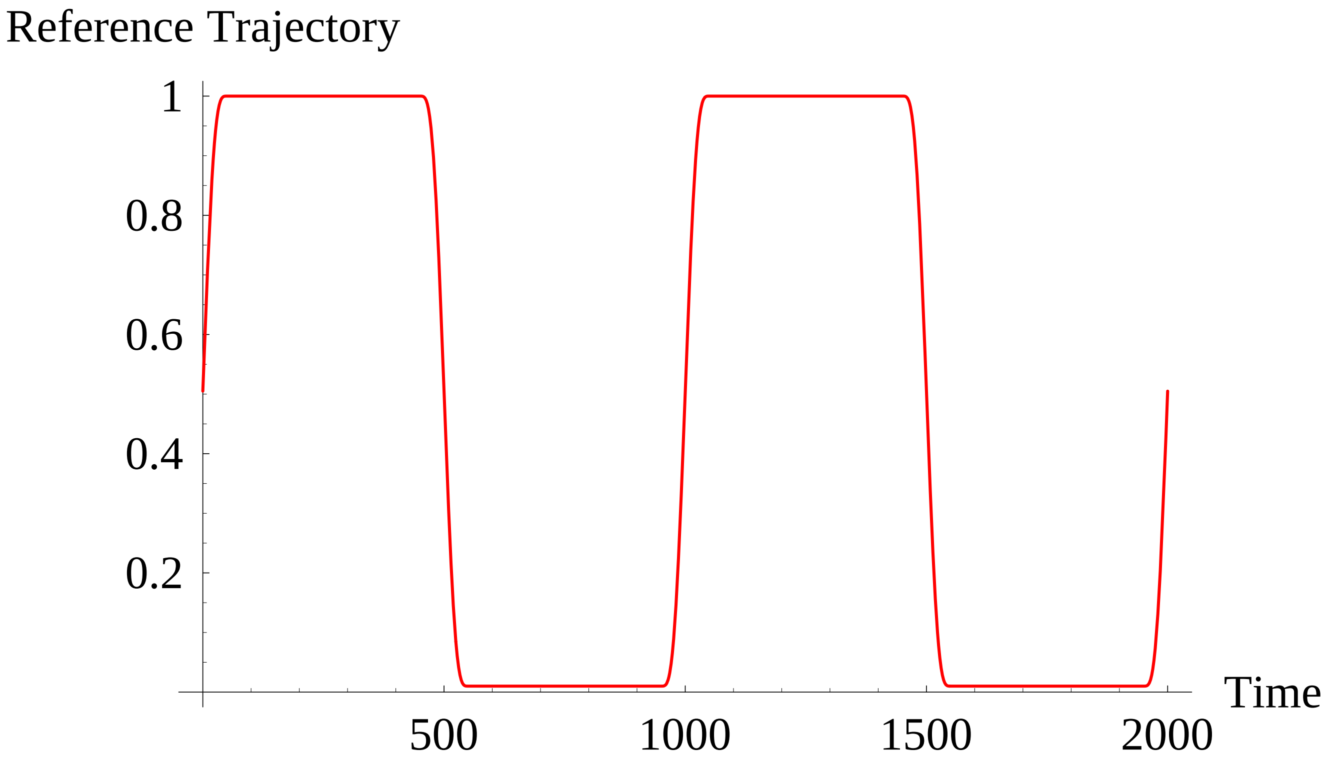
# CRAFTING the REFERENCE TRAJECTORY



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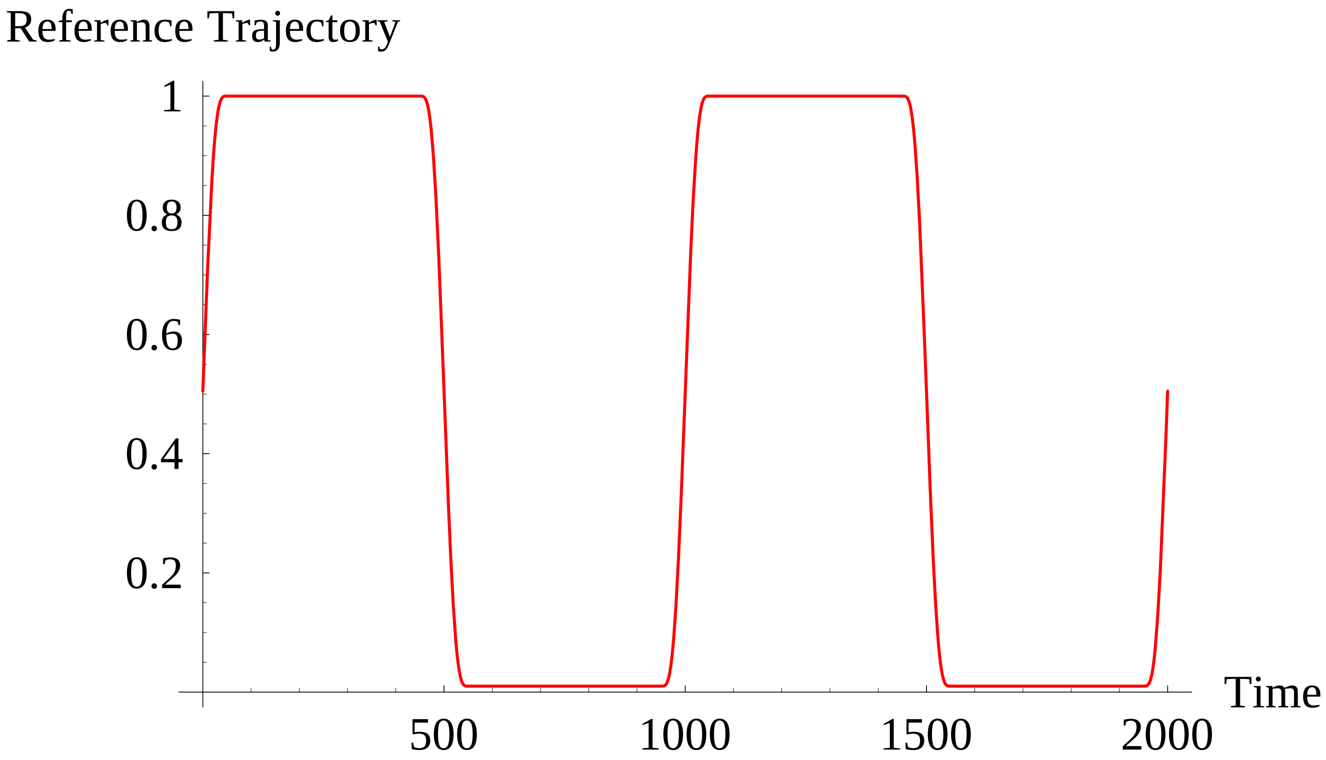



Mimics square wave with .01 offset.

# CRAFTING the REFERENCE TRAJECTORY

$$
\begin{aligned}
y_d(t) &= 0.01 + \varepsilon_1 \left[\mathcal{I}(500 + \min\{t, 50\})\right. \\
&\quad -\mathcal{I}(\min\{\max\{t, 450\}, 550\}) \\
&\quad \left. + \mathcal{I}(\max\{t, 950\} - 500)\right] \quad \text{for } \ 0 \le t \le 1000, \\
y_d(t) &= y_d(t - 1000) \ \ \text{for } \ t \ge 1000,
\end{aligned}
$$

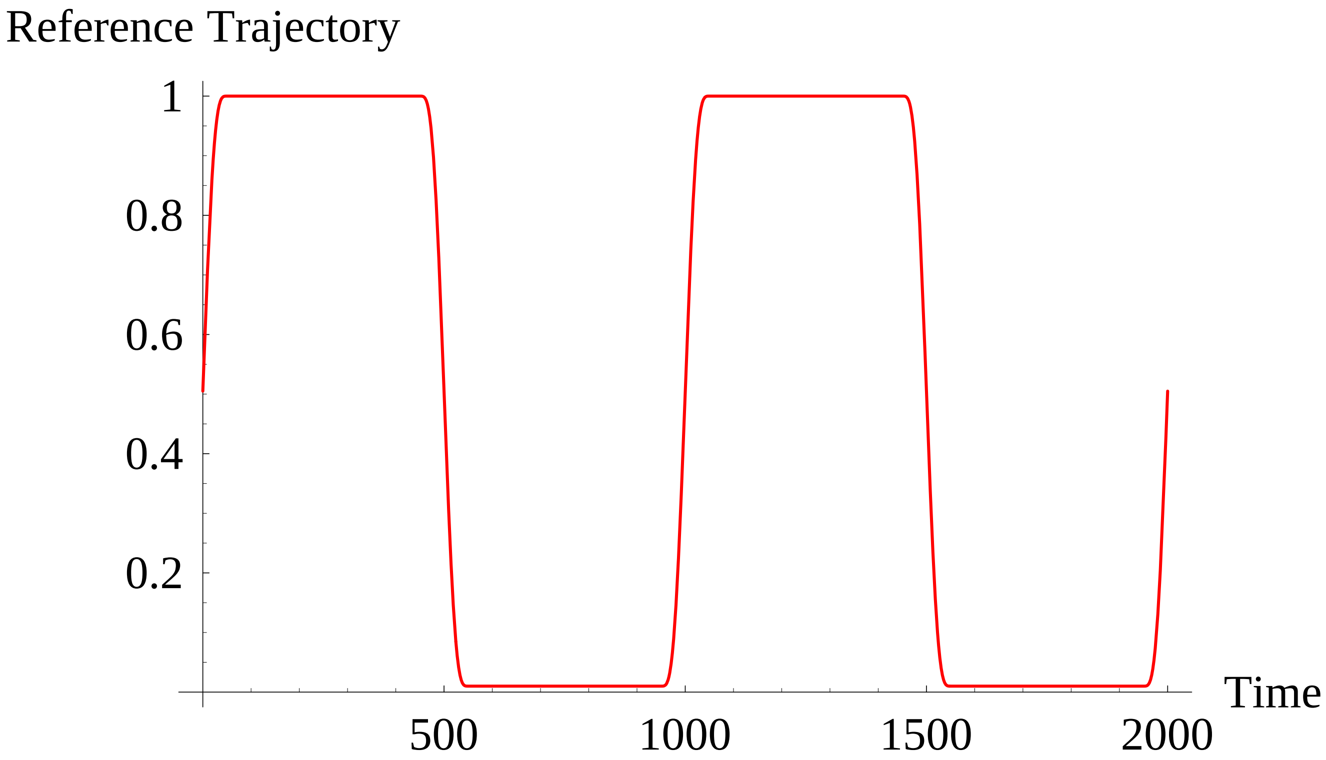
where

$$\mathcal{I}(r) = \int_{450}^{r} (s - 450)^3 (550 - s)^3 \mathrm{d}s$$

and

$$\varepsilon_1 = .99/\mathcal{I}(550).$$

# CRAFTING the REFERENCE TRAJECTORY

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# NUMERICAL VALIDATION

We chose

$$a_1 = a_2 = 1,$$

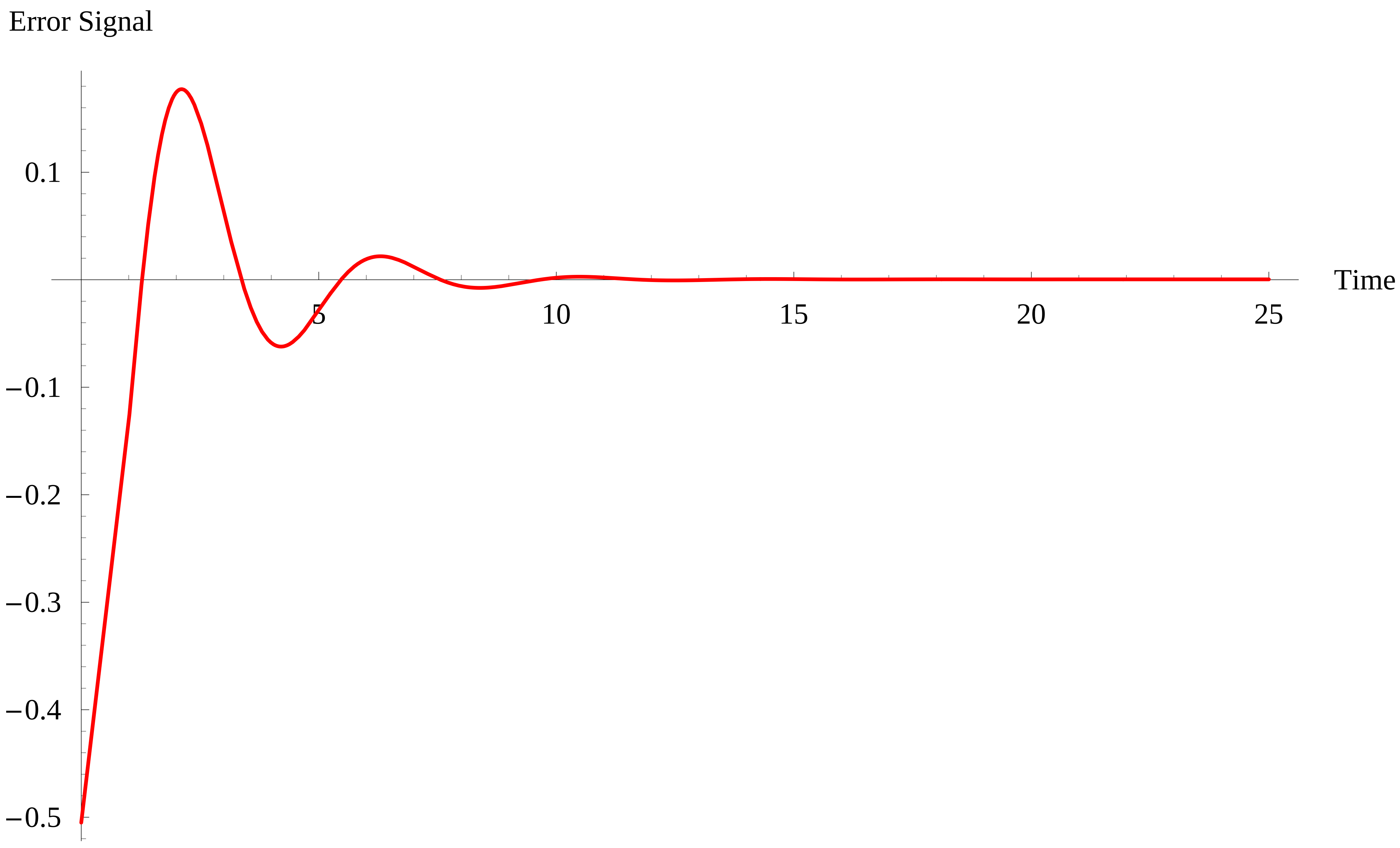
$$a_3 = 100,$$

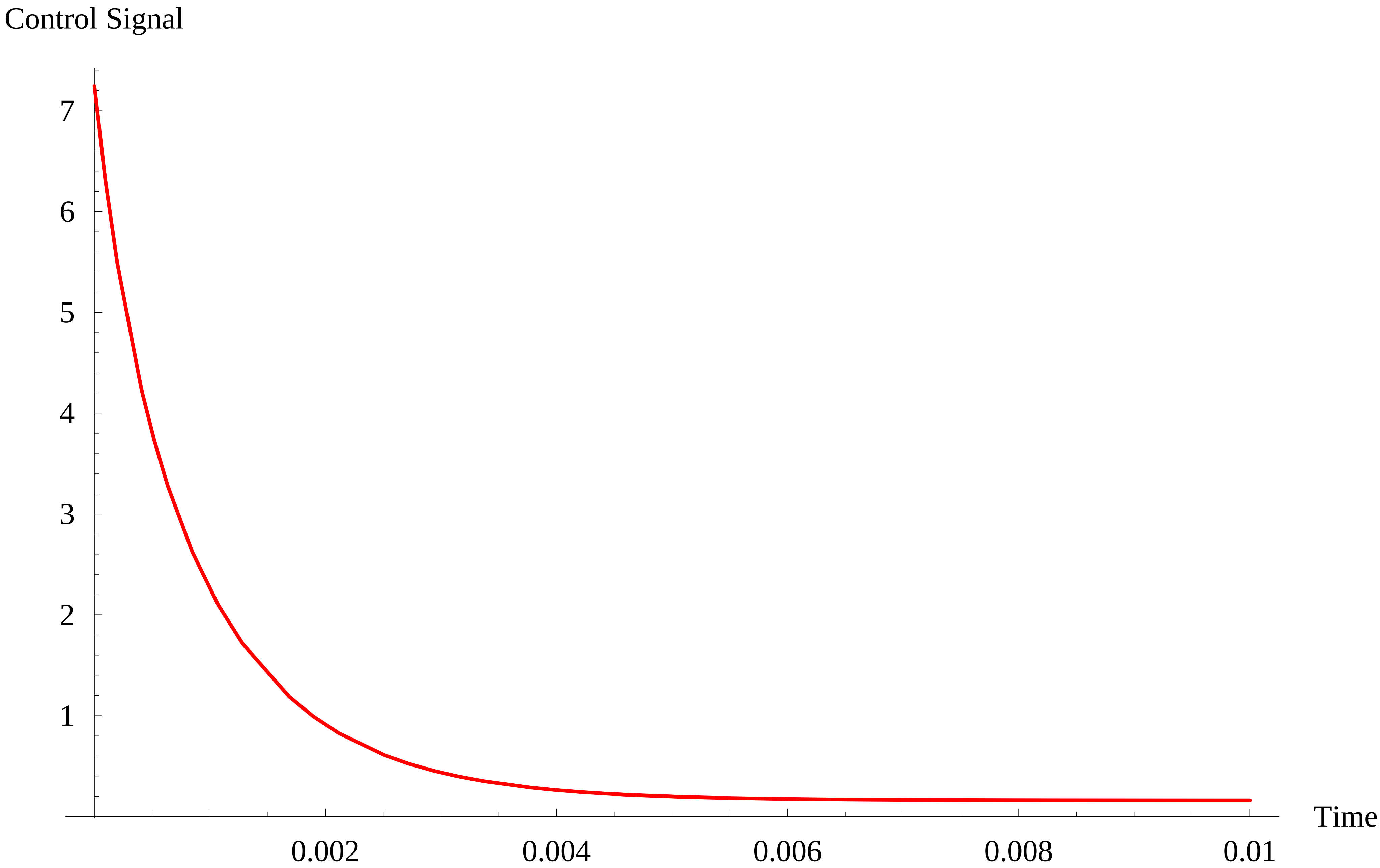
$$m = 1, \quad k = 2.5, \quad \gamma = 1,$$

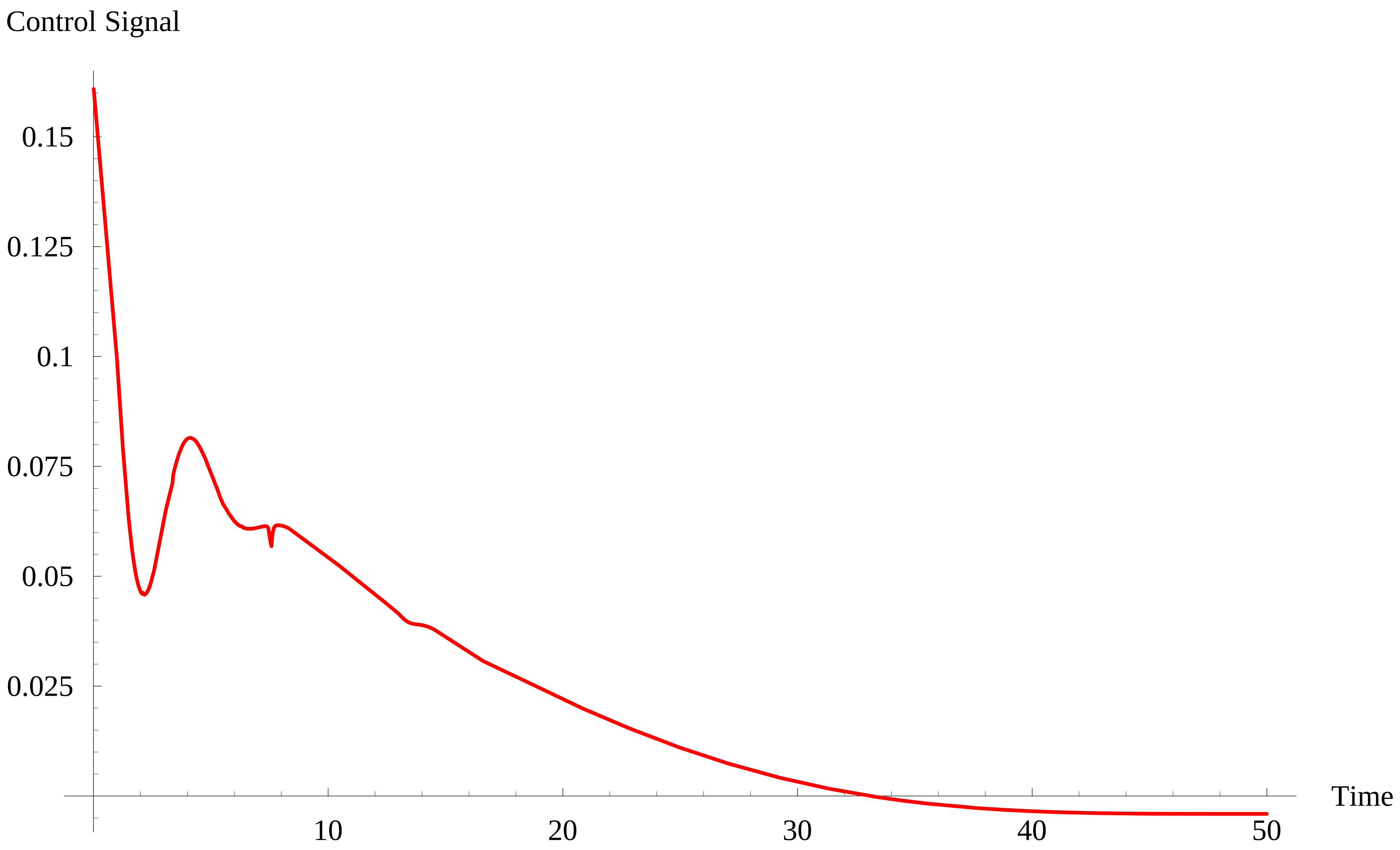
$$b = 1, \quad \alpha = 0.5, \quad \beta = 0.001,$$

$$g_o = 1,$$

$$(x, \dot{x}, z)(0) = (0, 0, 10).$$


Error Signal
0.1
−0.1
−0.2
−0.3
−0.4
−0.5
5
10
15
20
25
Time


Control Signal
7
6
5
4
3
2
1
Time
0.002
0.004
0.006
0.008
0.01



Control Signal
0.15
0.125
0.1
0.075
0.05
0.025
10
20
30
40
50
Time

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# CONCLUSIONS

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- We constructed a new tracking controller for MEM relays using a global strict Lyapunov function.
- We can prove ISS to additive uncertainty on $b$ and $k$ and compute explicit bounds on admissible disturbances.
- Extensions to partial state feedback with controllers independent of the velocity can be shown.
- For the proofs, see [MM., F. Mazenc, and M. de Queiroz, “Tracking and robustness analysis for controlled microelectromechanical relays,” *Intl. J. Robust Nonlinear Control*, to appear.]

## ACKNOWLEDGMENTS ETC.


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- For a .pdf with these slides, email malisoff at lsu dot edu.