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A QUADRATIC C^0 INTERIOR PENALTY METHOD FOR THE DISPLACEMENT OBSTACLE PROBLEM OF CLAMPED KIRCHHOFF PLATES*

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Abstract. We study a quadratic C^0 interior penalty method for the displacement obstacle problem of Kirchhoff plates with general Dirichlet boundary conditions on general polygonal domains. Under the conditions that the obstacles are sufficiently smooth and separated from each other and the boundary displacement, we prove that the magnitudes of the errors in the energy norm and the L_∞ norm are $O(h^\alpha)$, where h is the mesh size and $\alpha > \frac{1}{2}$ is determined by the interior angles of the polygonal domain. We also address the approximations of the coincidence set and the free boundary. The performance of the method is illustrated by numerical results.

Key words. displacement obstacle, clamped Kirchhoff plate, fourth order, variational inequality, free boundary, finite element, C^0 interior penalty method, discontinuous Galerkin method

AMS subject classifications. 65N30, 65K15, 74S05, 49J40

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1. Introduction.

Let $\Omega \subset \mathbb{R}^2$ be a bounded polygonal domain, $f(x) \in L_2(\Omega)$, $g(x) \in H^4(\Omega)$, and $\psi_1(x)$, $\psi_2(x) \in C^2(\Omega) \cap C(\bar{\Omega})$ such that

$$\psi_1 < \psi_2 \text{ in } \Omega \quad \text{and} \quad \psi_1 < g < \psi_2 \text{ on } \partial\Omega. \tag{1.1}$$

In this paper we consider the following two-sided displacement obstacle problem with nonhomogeneous Dirichlet boundary conditions: Find $u \in K$ such that

$$u = \underset{v \in K}{\operatorname{argmin}}\, G(v), \tag{1.2}$$

where

$$K = \{v \in H^2(\Omega) : v - g \in H_0^2(\Omega),\ \psi_1 \le v \le \psi_2 \text{ in } \Omega\}, \tag{1.3}$$

$$G(v) = \frac{1}{2}a(v,v) - (f,v), \tag{1.4}$$

$$a(w,v) = \int_\Omega D^2 w : D^2 v\, dx = \int_\Omega \sum_{i,j=1}^{2} \left(\frac{\partial^2 w}{\partial x_i \partial x_j}\right)\left(\frac{\partial^2 v}{\partial x_i \partial x_j}\right) dx, \tag{1.5}$$

and $(\cdot,\cdot)$ denotes the inner product of $L_2(\Omega)$.

Remark 1.1. When $g = 0$ the problem (1.2) is the mathematical model for the bending of a clamped thin plate that satisfies the Kirchhoff–Love hypothesis and is bounded between two obstacles, where Ω is the configuration domain, u is the vertical displacement of the midsurface, and f is the vertical load density divided by

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the flexural rigidity of the plate. Of course the results in this paper are also valid if there is only one obstacle (either from above or from below).

Note that K is a nonempty closed convex subset of $H^2(\Omega)$, and the symmetric bilinear form $a(\cdot,\cdot)$ is bounded on $H^2(\Omega)$ and coercive on the set $K-K=\{v-w: v,w\in K\}\subset H_0^2(\Omega)$. Therefore it follows from the standard theory in [40, 44, 31, 38] that the obstacle problem has a unique solution $u\in K$ which is also uniquely determined by the variational inequality

$$a(u,v-u)\geq (f,v-u) \qquad \forall\, v\in K. \tag{1.6}$$

It was shown in [29, 30, 22, 45, 31, 16] that the solution u of (1.2) belongs to $H^3_{loc}(\Omega)\cap C^2(\Omega)$ under our assumptions on f, ψ_1, ψ_2, and g. In view of (1.1), u is unconstrained near $\partial\Omega$, and hence $\Delta^2 u=f$ in a neighborhood of $\partial\Omega$, which implies (cf. [3, 33, 25, 39]) that $u\in H^{2+\alpha}(\Omega)$ for some $\alpha\in(\frac{1}{2},1]$ determined by the interior angles of Ω. We will refer to α as the index of elliptic regularity. In the case where Ω is convex, we can take α to be 1.

Since in general the solution u of (1.2) does not belong to $H^4_{loc}(\Omega)$, and $\Delta^2 u-f$ is only a measure [22], the complementarity form of the variational inequality (1.6) exists only in a weak sense. Therefore the techniques developed in [28, 20, 23, 32, 46, 36, 47] for second order elliptic obstacle problems lead to suboptimal error estimates for finite element methods for (1.2).

A new unified convergence analysis for conforming finite element methods, nonconforming finite element methods, and discontinuous Galerkin methods for (1.2) was developed in [16] for convex polygonal domains and homogeneous Dirichlet boundary conditions. The magnitude of the discretization error in the energy norm was shown to be $O(h)$, which is optimal. A key ingredient in the error analysis is an auxiliary obstacle problem that connects the continuous and discrete obstacle problems.

The goal of this paper is to extend the result in [16] to general polygonal domains and general Dirichlet boundary conditions for a quadratic C^0 interior penalty method. We will show that the magnitudes of the discretization errors in the energy norm and the L_∞ norm are $O(h^\alpha)$, and that the discrete free boundaries converge to the continuous free boundary under appropriate assumptions.

The rest of the paper is organized as follows. In section 2 we introduce the quadratic C^0 interior penalty method and recall some results that are useful for the convergence analysis, which is carried out in section 3. In section 4 we derive an L_∞ error estimate and discuss the approximations of the coincidence set and the free boundary. Numerical examples are presented in section 5 to illustrate the performance of the quadratic C^0 interior penalty method. We end with some concluding remarks in section 6.

2. A quadratic C^0 interior penalty method.

In this section we define a quadratic C^0 interior penalty method for (1.2) and collect some results that will be used in section 3.

2.1. Preliminaries.

Let $\mathcal{T}_h$ be a regular triangulation of Ω with mesh size h and let $V_h\,(\subset H^1(\Omega))$ be the P_2 Lagrange finite element space associated with $\mathcal{T}_h$. We will denote the set of the edges of the triangles in $\mathcal{T}_h$ by $\mathcal{E}_h=\mathcal{E}_h^i\cup\mathcal{E}_h^b$, where $\mathcal{E}_h^i$ is the set of the edges interior to Ω and $\mathcal{E}_h^b$ is the set of the edges on $\partial\Omega$. We will also use $\mathcal{T}_e$ to denote the set of (one or two) triangles in $\mathcal{T}_h$ that share e as a common edge. The diameter of a triangle T (resp., an edge e) will be denoted by h_T (resp., $|e|$). Up to a multiplicative constant the mesh parameter h is equivalent to $\max_{T\in\mathcal{T}_h} h_T$.

The construction and analysis of the quadratic C^0 interior penalty method require the concepts of jumps and means of normal derivatives across the edges in $\mathcal{T}_h$, which are defined for functions in the piecewise Sobolev space

$$H^s(\Omega,\mathcal{T}_h)=\{v\in L_2(\Omega):\ v_T=v\big|_T\in H^s(T)\quad\forall\, T\in\mathcal{T}_h\}.$$

Let $e\in\mathcal{E}_h^i$, let $\mathcal{T}_e=\{T_-,T_+\}$, and let n_e be the unit normal of e pointing from T_- to T_+. We define on e

$$\left\{\!\!\left\{\frac{\partial^2 v}{\partial n^2}\right\}\!\!\right\}=\frac{1}{2}\left(\frac{\partial^2 v_-}{\partial n_e^2}+\frac{\partial^2 v_+}{\partial n_e^2}\right)\qquad\forall\, v\in H^s(\Omega,\mathcal{T}_h),\ s>\frac{5}{2},$$

and

$$\left[\!\!\left[\frac{\partial v}{\partial n}\right]\!\!\right]=\frac{\partial v_+}{\partial n_e}-\frac{\partial v_-}{\partial n_e}\qquad\qquad\forall\, v\in H^2(\Omega,\mathcal{T}_h),$$

where $v_\pm=v\big|_{T_\pm}$. Note that $\{\!\{\partial^2 v/\partial n^2\}\!\}$ and $[\![\partial v/\partial n]\!]$, which appear in the definition of the C^0 interior penalty method, are independent of the choice of $T_\pm$. For the analysis of the C^0 interior penalty method we also need

$$\left[\!\!\left[\frac{\partial^2 v}{\partial n_e^2}\right]\!\!\right]=\frac{\partial^2 v_+}{\partial n_e^2}-\frac{\partial^2 v_-}{\partial n_e^2},\quad\left[\!\!\left[\frac{\partial^2 v}{\partial n_e\partial t_e}\right]\!\!\right]=\frac{\partial^2 v_+}{\partial n_e\partial t_e}-\frac{\partial^2 v_-}{\partial n_e\partial t_e}\quad\forall\, v\in H^s(\Omega,\mathcal{T}_h),\ s>\frac{5}{2},$$

and

$$\left\{\!\!\left\{\frac{\partial v}{\partial n_e}\right\}\!\!\right\}=\frac{1}{2}\left(\frac{\partial v_+}{\partial n_e}+\frac{\partial v_-}{\partial n_e}\right)\qquad\qquad\forall\, v\in H^2(\Omega,\mathcal{T}_h),$$

where the unit tangent vector t_e is obtained by rotating n_e by a counterclockwise right-angle. These concepts do depend on the choices of $T_\pm$.

On $e\in\mathcal{E}_h^b$ with $\mathcal{T}_e=\{T\}$, we take n_e to be the unit normal of e that points towards the outside of Ω and define

$$\left\{\!\!\left\{\frac{\partial^2 v}{\partial n^2}\right\}\!\!\right\}=\frac{\partial^2 v_T}{\partial n_e^2}\quad\forall\, v\in H^s(\Omega,\mathcal{T}_h)\ (s>5/2)\ \text{and}\ \left[\!\!\left[\frac{\partial v}{\partial n}\right]\!\!\right]=-\frac{\partial v_T}{\partial n_e}\quad\forall\, v\in H^2(\Omega,\mathcal{T}_h),$$

where $v_T=v\big|_T$.

Let $T\in\mathcal{T}_h$, $v\in P_2(T)$, and $w\in H^2(T)$. We have an integration by parts formula

$$\int_T D^2v:D^2w\,dx=\int_{\partial T}\left[\left(\frac{\partial^2 v}{\partial n^2}\right)\left(\frac{\partial w}{\partial n}\right)+\left(\frac{\partial^2 v}{\partial n\partial t}\right)\left(\frac{\partial w}{\partial t}\right)\right]ds,$$

where $\partial/\partial n$ (resp., $\partial/\partial t$) denotes the outward normal derivative (resp., counterclockwise tangential derivative) along ∂T.

By summing up the integration by parts formula over all the triangles in $\mathcal{T}_h$, we find

$$\sum_{T\in\mathcal{T}_h}\int_T D^2v:D^2w\,dx=-\sum_{e\in\mathcal{E}_h^i}\int_e\left(\left[\!\!\left[\frac{\partial^2 v}{\partial n_e^2}\right]\!\!\right]\left\{\!\!\left\{\frac{\partial w}{\partial n_e}\right\}\!\!\right\}+\left[\!\!\left[\frac{\partial^2 v}{\partial n_e\partial t_e}\right]\!\!\right]\frac{\partial w}{\partial t_e}\right)ds$$
$$-\sum_{e\in\mathcal{E}_h}\int_e\left\{\!\!\left\{\frac{\partial^2 v}{\partial n^2}\right\}\!\!\right\}\left[\!\!\left[\frac{\partial w}{\partial n}\right]\!\!\right]ds\tag{2.1}$$

for all $v\in V_h$ and $w\in H^2(\Omega;\mathcal{T}_h)\cap H_0^1(\Omega)$.

2.2. The discrete obstacle problem. Let $\mathcal{V}_h$ be the set of the vertices of the triangles in $\mathcal{T}_h$. The C^0 interior penalty method for (1.2) is to find $u_h \in K_h$ such that

$$u_h = \operatorname*{argmin}_{v\in K_h} G_h(v), \tag{2.2}$$

where

$$G_h(v) = \frac{1}{2}\bigg(\sum_{T\in\mathcal{T}_h}\int_T D^2v : D^2v\,dx + 2\sum_{e\in\mathcal{E}_h}\int_e \{\!\{\partial^2 v/\partial n^2\}\!\}[\![\partial(v-g)/\partial n]\!]ds + \sigma\sum_{e\in\mathcal{E}_h}|e|^{-1}\int_e [\![\partial(v-g)/\partial n]\!][\![\partial(v-g)/\partial n]\!]ds\bigg) - (f,v),$$

$$K_h = \{v\in V_h : v - \Pi_h g \in H_0^1(\Omega),\ \psi_1(p)\le v(p)\le \psi_2(p)\quad \forall\, p\in\mathcal{V}_h\}, \tag{2.3}$$

and σ is a positive penalty parameter.

Note that

$$(\Pi_h\zeta)(p) = \zeta(p) \qquad \forall\, p\in\mathcal{V}_h,\ \zeta\in C(\bar\Omega), \tag{2.4}$$

where $\Pi_h : C(\bar\Omega) \longrightarrow V_h$ is the nodal interpolation operator for the P_2 Lagrange finite element space. In particular we have $\Pi_h K \subset K_h$, and hence K_h is nonempty.

Remark 2.1. While the approximation of the boundary condition $u = g$ is included in the definition of K_h, the approximation of the boundary condition $\partial u/\partial n = \partial g/\partial n$ is enforced by the penalty term in $G_h(\cdot)$.

Remark 2.2. The functional $G_h(\cdot)$ is motivated by the bilinear form for C^0 interior penalty methods for the biharmonic equation with general Dirichlet boundary conditions. C^0 interior penalty methods for fourth order elliptic boundary value problems were introduced in [27] and further investigated in [14, 17, 15, 10, 11, 18] and [8, 12, 9]. Other C^0 discontinuous Galerkin methods for fourth order problems are discussed in [48, 37]. C^0 interior penalty methods have certain advantages over classical finite element methods for fourth order problems, such as their simplicity and symmetric positive-definiteness, the existence of isoparametric C^0 interior penalty methods for curved domains, and the existence of natural preconditioners. Further details on the comparison of C^0 interior penalty methods with classical finite element methods can be found in [27, 15, 8, 12].

Since $[\![\partial g/\partial n]\!] = 0$ on interior edges, we can reformulate (2.2) as follows: Find $u_h \in K_h$ such that

$$u_h = \operatorname*{argmin}_{v\in K_h}\left[\frac{1}{2}a_h(v,v) - F(v)\right], \tag{2.5}$$

where

$$a_h(w,v) = \sum_{T\in\mathcal{T}_h}\int_T D^2w : D^2v\,dx + \sum_{e\in\mathcal{E}_h}\int_e \{\!\{\partial^2 w/\partial n^2\}\!\}[\![\partial v/\partial n]\!]ds + \sum_{e\in\mathcal{E}_h}\int_e \{\!\{\partial^2 v/\partial n^2\}\!\}[\![\partial w/\partial n]\!]ds + \sigma\sum_{e\in\mathcal{E}_h}|e|^{-1}\int_e [\![\partial w/\partial n]\!][\![\partial v/\partial n]\!]ds, \tag{2.6}$$

$$\begin{aligned} F(v) &= (f,v) + \sum_{e\in\mathcal{E}_h^b}\int_e\left(\left\{\!\!\left\{\frac{\partial^2 v}{\partial n^2}\right\}\!\!\right\} + \frac{\sigma}{|e|}\left[\!\!\left[\frac{\partial v}{\partial n}\right]\!\!\right]\right)\left[\!\!\left[\frac{\partial g}{\partial n}\right]\!\!\right]ds \\ &= (f,v) + \sum_{e\in\mathcal{E}_h^b}\int_e\left(\left\{\!\!\left\{\frac{\partial^2 v}{\partial n^2}\right\}\!\!\right\} + \frac{\sigma}{|e|}\left[\!\!\left[\frac{\partial v}{\partial n}\right]\!\!\right]\right)\left[\!\!\left[\frac{\partial u}{\partial n}\right]\!\!\right]ds. \end{aligned} \tag{2.7}$$

We will measure the discretization error by the energy seminorm $\|\cdot\|_h$ defined by

$$\|v\|_h^2 = \sum_{T\in\mathcal{T}_h} |v|_{H^2(T)}^2 + \sum_{e\in\mathcal{E}_h} |e|\,\|\{\!\{\partial^2 v/\partial n^2\}\!\}\|_{L_2(e)}^2 + \sum_{e\in\mathcal{E}_h} |e|^{-1}\|[\![\partial v/\partial n]\!]\|_{L_2(e)}^2. \tag{2.8}$$

Remark 2.3. The seminorm $\|\cdot\|_h$ is well defined on the space $V_h + H^{2+\alpha}(\Omega)$, where $\alpha \in (\frac{1}{2}, 1]$ is the index of elliptic regularity. Moreover, it is a norm on the space $V_h \cap H_0^1(\Omega) \supset (K_h - K_h)$.

The bilinear form $a_h(\cdot,\cdot)$ is coercive with respect to $\|\cdot\|_h$ (cf. [14]); i.e.,

$$a_h(v,v) \geq C\|v\|_h^2 \qquad \forall\, v \in V_h, \tag{2.9}$$

provided that σ is sufficiently large, which is assumed to be the case. (It is known from [14, 9] that σ can be taken to be 5.)

Remark 2.4. From here on, we will use C (with or without a subscript) to denote a generic positive constant independent of h.

From Remark 2.3 and (2.9) we see that $a_h(\cdot,\cdot)$ is positive definite on the set $K_h - K_h\,(\subset V_h \cap H_0^1(\Omega))$. Therefore it follows from the standard theory that the discrete obstacle problem (2.2)/(2.5) has a unique solution characterized by the discrete variational inequality

$$a_h(u_h, v - u_h) \geq F(v - u_h) \qquad \forall v \in K_h. \tag{2.10}$$

Below we collect some useful properties of the nodal interpolation operator Π_h. The following estimates (cf. [23, 13]) are standard:

$$\sum_{m=0}^{2} h_T^m |\zeta - \Pi_h\zeta|_{H^m(T)} \leq C h_T^{2+\alpha}|\zeta|_{H^{2+\alpha}(T)} \qquad \forall \zeta \in H^{2+\alpha}(\Omega),\ T \in \mathcal{T}_h. \tag{2.11}$$

We can use (2.11) and the trace theorem with scaling to derive an interpolation error estimate (cf. [14])

$$\|\zeta - \Pi_h\zeta\|_h \leq C h^\alpha |\zeta|_{H^{2+\alpha}(\Omega)} \qquad \forall \zeta \in H^{2+\alpha}(\Omega). \tag{2.12}$$

In particular, we have

$$\begin{aligned}\sum_{e\in\mathcal{E}_h^i} |e|^{-1}\|[\![\partial(\Pi_h\zeta)/\partial n]\!]\|_{L_2(e)}^2 &= \sum_{e\in\mathcal{E}_h^i} |e|^{-1}\|[\![\partial(\zeta - \Pi_h\zeta)/\partial n]\!]\|_{L_2(e)}^2 \\ &\leq \|\zeta - \Pi_h\zeta\|_h^2 \leq C h^{2\alpha}|\zeta|_{H^{2+\alpha}(\Omega)}^2 \qquad \forall\, \zeta \in H^{2+\alpha}(\Omega).\end{aligned} \tag{2.13}$$

The following result is useful for the convergence analysis.

LEMMA 2.5. *We have*

$$\sum_{e\in\mathcal{E}_h^i} |e|\|[\![\partial^2(\Pi_h\zeta)/\partial n_e^2]\!]\|_{L_2(e)}^2 \leq C h^{2\alpha}|\zeta|_{H^{2+\alpha}(\Omega)}^2 \qquad \forall\, \zeta \in H^{2+\alpha}(\Omega).$$

Proof. Let $e \in \mathcal{E}_h^i$ be arbitrary and Q_e be the quadrilateral formed by the two triangles in $\mathcal{T}_e$. By the Bramble–Hilbert lemma [4, 26], there exists $z_e \in P_2(Q_e)$ such that $|\zeta - z_e|_{H^2(Q_e)} \leq C|e|^\alpha|\zeta|_{H^{2+\alpha}(Q_e)}$. Combining this estimate with (2.11) and a standard inverse estimate [23, 13], we find

$$\begin{aligned}\sum_{e\in\mathcal{E}_h^i}|e|\|[\![\partial^2(\Pi_h\zeta)/\partial n_e^2]\!]\|_{L_2(e)}^2 &= \sum_{e\in\mathcal{E}_h^i}|e|\|[\![\partial^2(z_e-\Pi_h\zeta)/\partial n_e^2]\!]\|_{L_2(e)}^2\\ &\le C\sum_{e\in\mathcal{E}_h^i}\sum_{T\in\mathcal{T}_e}|z_e-\Pi_h\zeta|_{H^2(T)}^2\\ &\le C\sum_{e\in\mathcal{E}_h^i}\sum_{T\in\mathcal{T}_e}\big(|z_e-\zeta|_{H^2(T)}^2+|\zeta-\Pi_h\zeta|_{H^2(T)}^2\big)\\ &\le Ch^{2\alpha}\sum_{e\in\mathcal{E}_h^i}\sum_{T\in\mathcal{T}_e}|\zeta|_{H^{2+\alpha}(Q_e)}^2\le Ch^{2\alpha}|\zeta|_{H^{2+\alpha}(\Omega)}^2. \qquad \square\end{aligned}$$

2.3. An auxiliary obstacle problem. We can connect the continuous obstacle problem (1.2) and the discrete obstacle problem (2.2)/(2.5) by an intermediate obstacle problem: Find $\tilde{u}_h \in \tilde{K}_h$ such that

$$\tilde{u}_h = \operatorname*{argmin}_{v\in\tilde{K}_h} G(v), \tag{2.14}$$

where G is given by (1.4)–(1.5) and

$$\tilde{K}_h = \{v\in H^2(\Omega) : v-g\in H_0^2(\Omega),\ \psi_1(p)\le v(p)\le\psi_2(p)\quad \forall p\in\mathcal{V}_h\}. \tag{2.15}$$

Note that $\tilde{K}_h$ is a closed convex subset of $H^2(\Omega)$ and $K\subset\tilde{K}_h$. The unique solution of (2.14) is characterized by the variational inequality:

$$a(\tilde{u}_h, v-\tilde{u}_h)\ge(f, v-\tilde{u}_h)\qquad \forall v\in\tilde{K}_h. \tag{2.16}$$

The connection between (1.2) and (2.14) is given by the following properties of $\tilde{u}_h$ from [16, section 3]:

$$|u-\tilde{u}_h|_{H^2(\Omega)}\le Ch, \tag{2.17}$$

and there exists $h_0>0$ such that

$$\hat{u}_h=\tilde{u}_h+\delta_{h,1}\phi_1-\delta_{h,2}\phi_2\in K\qquad \forall\, h\le h_0, \tag{2.18}$$

where $\phi_1,\phi_2\in C_0^\infty(\Omega)$ and the positive numbers $\delta_{h,1}$ and $\delta_{h,2}$ satisfy

$$\delta_{h,i}\le Ch^2. \tag{2.19}$$

Remark 2.6. Even though we consider only obstacle problems for convex domains and homogeneous Dirichlet boundary conditions in [16], the relations (2.17)–(2.19) remain valid for general polygonal domains and general Dirichlet boundary conditions because the arguments in [16, section 3] require only (i) $\tilde{K}_h$ is a closed convex subset of $H^2(\Omega)$, (ii) $K\subset\tilde{K}_h$, (iii) the separation of the obstacle functions and the boundary displacement (cf. (1.1)), (iv) the smoothness assumptions on the obstacle functions, and (v) $u\in C^2(\Omega)$.

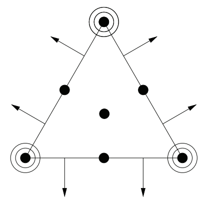


FIG. 2.1. *Degrees of freedom for the P_6 Argyris element.*

2.4. Enriching operator. We can connect (2.14) and (2.2)/(2.5) through a linear operator

$$E_h : V_h \cap H_0^1(\Omega) \longrightarrow \tilde{V}_h \cap H_0^2(\Omega), \tag{2.20}$$

where $\tilde{V}_h$ is the P_6 Argyris finite element space (cf. [2]) associated with $\mathcal{T}_h$.

The enriching operator E_h is constructed by averaging as follows. For $v \in V_h \cap H_0^1(\Omega)$, we define $E_h v \in \tilde{V}_h \cap H_0^2(\Omega)$ by specifying its degrees of freedom (dofs) (cf. Figure 2.1), which are (i) the values of the derivatives of $E_h v$ up to second order at the vertices of $\mathcal{T}_h$, (ii) the values of $E_h v$ at the midpoints of the edges of $\mathcal{T}_h$ and at the centers of the triangles of $\mathcal{T}_h$, and (iii) the values of the normal derivatives of $E_h v$ at two nodes on each edge of $\mathcal{T}_h$. The dofs of $E_h v$ at any node (with the exception of the dofs along $\partial\Omega$ that involve differentiation) are defined to be the average of the corresponding dofs of v from the triangles of $\mathcal{T}_h$ that share the node. Since v is continuous at the vertices, midpoints, and centers, $E_h v = v$ at these nodes. In particular we have

$$(E_h v)(p) = v(p) \qquad \forall\, p \in \mathcal{V}_h. \tag{2.21}$$

To ensure that $E_h v \in H_0^2(\Omega)$, we take the normal derivative of $E_h v$ at the nodes on the boundary edges to be 0. Similarly we assign the value 0 to all first order derivatives of $E_h v$ at the vertices on $\partial\Omega$, and at a corner of Ω we also assign the value 0 to all the second order derivatives of $E_h v$. Finally we define $\partial^2(E_h v)/\partial t^2$ and $\partial^2(E_h v)/\partial t \partial n$ to be 0 at the vertices on $\partial\Omega$ that are not one of the corners of Ω, and we define the remaining second order derivative $\partial^2(E_h v)/\partial n^2$ at such a vertex by averaging. (Here $\partial/\partial t$ and $\partial/\partial n$ are the differentiation in the tangential and normal directions along $\partial\Omega$, respectively.)

The proof of the following properties of E_h can be found in [14]. Letting $T \in \mathcal{T}_h$ be arbitrary, we have

$$\sum_{m=0}^{2} h_T^{2m} |v - E_h v|_{H^m(T)}^2 \le C h_T^4 \Bigg(\sum_{T' \in \mathcal{T}_T} |v|_{H^2(T')}^2 + \sum_{e \in \mathcal{E}_{\mathcal{V}(T)}} |e|^{-1} \| [\![\partial v/\partial n]\!] \|_{L_2(e)}^2 \Bigg) \tag{2.22}$$

for any $v \in V_h \cap H_0^1(\Omega)$, where $\mathcal{T}_T$ is the set of the triangles in $\mathcal{T}_h$ sharing a common vertex with T and $\mathcal{E}_{\mathcal{V}(T)}$ is the set of the edges in $\mathcal{T}_T$ sharing a vertex with T, and also

$$\sum_{m=0}^{2} h_T^m |\zeta - E_h \Pi_h \zeta|_{H^m(T)} \le C h_T^{2+\alpha} |\zeta|_{H^{2+\alpha}(S_T)} \quad \forall\, \zeta \in H^{2+\alpha}(\Omega) \cap H_0^2(\Omega), \tag{2.23}$$

where S_T is the union of the triangles in $\mathcal{T}_T$.

It follows from (2.8), (2.22), (2.23), and the trace theorem with scaling that

$$\|v - E_h v\|_{L_2(\Omega)} + h|v - E_h v|_{H^1(\Omega)} + h^2|E_h v|_{H^2(\Omega)} \le Ch^2\|v\|_h, \tag{2.24}$$

$$\sum_{e\in\mathcal{E}_h^i} |e|^{-1}\|\{\!\{\partial(v - E_h v)/\partial n_e\}\!\}\|^2_{L_2(e)} \le C\|v\|_h^2, \tag{2.25}$$

$$\sum_{e\in\mathcal{E}_h^i} |e|^{-1}\|\partial(v - E_h v)/\partial t_e\|^2_{L_2(e)} \le C\|v\|_h^2 \tag{2.26}$$

for all $v \in V_h \cap H_0^1(\Omega)$, and

$$\sum_{m=0}^{2} h^m|\zeta - E_h\Pi_h\zeta|_{H^m(\Omega)} \le Ch^{2+\alpha}|\zeta|_{H^{2+\alpha}(\Omega)}, \tag{2.27}$$

$$\sum_{e\in\mathcal{E}_h^i} |e|^{-1}\|\partial(\zeta - E_h\Pi_h\zeta)/\partial n_e\|^2_{L_2(e)} \le Ch^{2\alpha}|\zeta|^2_{H^{2+\alpha}(\Omega)}, \tag{2.28}$$

$$\sum_{e\in\mathcal{E}_h^i} |e|^{-1}\|\partial(\zeta - E_h\Pi_h\zeta)/\partial t_e\|^2_{L_2(e)} \le Ch^{2\alpha}|\zeta|^2_{H^{2+\alpha}(\Omega)} \tag{2.29}$$

for all $\zeta \in H^{2+\alpha}(\Omega) \cap H_0^2(\Omega)$.

Remark 2.7. Enriching operators were first introduced in the context of domain decomposition algorithms for nonconforming finite element methods [5, 7, 6].

Since $v - \Pi_h g \in V_h \cap H_0^1(\Omega)$ for $v \in K_h$, we can define an operator $T_h : K_h \longrightarrow H^2(\Omega)$ by

$$T_h v = g + E_h(v - \Pi_h g) \qquad \forall v \in K_h. \tag{2.30}$$

The following properties of T_h are useful for the convergence analysis.

LEMMA 2.8. *We have*

$$T_h : K_h \longrightarrow \tilde{K}_h, \tag{2.31}$$

and, for any $v \in K_h$ *and* $\zeta \in H^{2+\alpha}(\Omega) \cap K$,

$$|T_h\Pi_h\zeta - T_h v|_{H^2(\Omega)} \le C\|\Pi_h\zeta - v\|_h, \tag{2.32}$$

$$\sum_{m=0}^{2} h^m|\zeta - T_h\Pi_h\zeta|_{H^m(\Omega)} \le Ch^{2+\alpha}|\zeta - g|_{H^{2+\alpha}(\Omega)}. \tag{2.33}$$

Proof. Let $v \in K_h$. Note that $T_h v - g \in H_0^2(\Omega)$ by (2.3) and (2.20), and

$$(T_h v)(p) = g(p) + [E_h(v - \Pi_h g)](p) = g(p) + (v - \Pi_h g)(p) = v(p) \qquad \forall\, p \in \mathcal{V}_h$$

by (2.4) and (2.21). It then follows (2.3) and (2.15) that $T_h v \in \tilde{K}_h$.

From (2.30) we have

$$T_h\Pi_h\zeta - T_h v = E_h(\Pi_h\zeta - v), \tag{2.34}$$

$$\zeta - T_h\Pi_h\zeta = (\zeta - g) - E_h\Pi_h(\zeta - g). \tag{2.35}$$

Since $\Pi_h\zeta - v \in K_h - K_h \subset V_h \cap H_0^1(\Omega)$ and $\zeta - g \in H^{2+\alpha}(\Omega) \cap H_0^2(\Omega)$, the properties (2.32) and (2.33) follow from (2.24) and (2.27). ∎

3. Convergence analysis. We begin with a simple estimate

$$\|\Pi_h u - u_h\|_h^2 \le Ca(\Pi_h u - u_h, \Pi_h u - u_h) \le C[a_h(\Pi_h u, \Pi_h u - u_h) - F(\Pi_h u - u_h)] \tag{3.1}$$

that follows from (2.9), (2.10), and the fact that $\Pi_h u \in K_h$. In view of (2.12), it remains only to estimate the right-hand side of (3.1).

The following two technical lemmas will be needed.

LEMMA 3.1. *We have*

$$|a(u, T_h\Pi_h u - u)| \le Ch^{2\alpha}.$$

Proof. First we use (1.5) and (2.35) to write

$$\begin{aligned} a(u, T_h\Pi_h u - u) &= a\big(u, (g-u) - E_h\Pi_h(g-u)\big) \\ &= \sum_{T\in\mathcal{T}_h} \int_T D^2(\Pi_h u) : D^2[(g-u) - E_h\Pi_h(g-u)]dx \\ &\quad + \sum_{T\in\mathcal{T}_h} \int_T D^2(u - \Pi_h u) : D^2[(g-u) - E_h\Pi_h(g-u)]dx, \end{aligned} \tag{3.2}$$

and we note that (2.8), (2.12), and (2.27) imply

$$\begin{aligned} &\left| \sum_{T\in\mathcal{T}_h} \int_T D^2(u - \Pi_h u) : D^2[(g-u) - E_h\Pi_h(g-u)]dx \right| \\ &\qquad \le \left(\sum_{T\in\mathcal{T}_h} |u - \Pi_h u|_{H^2(T)}^2 \right)^{\frac{1}{2}} |(g-u) - E_h\Pi_h(g-u)|_{H^2(\Omega)} \le Ch^{2\alpha}. \end{aligned} \tag{3.3}$$

Since $g-u$ and $E_h\Pi_h(g-u)$ belong to $H_0^2(\Omega)$ by (1.3) and (2.20), we can apply (2.1) to write

$$\begin{aligned} &\sum_{T\in\mathcal{T}_h} \int_T D^2(\Pi_h u) : D^2[(g-u) - E_h\Pi_h(g-u)]dx \\ &\qquad = -\sum_{e\in\mathcal{E}_h^i} \int_e \left[\!\!\left[\frac{\partial^2(\Pi_h u)}{\partial n_e^2} \right]\!\!\right] \frac{\partial[(g-u) - E_h\Pi_h(g-u)]}{\partial n_e} ds \\ &\qquad\quad - \sum_{e\in\mathcal{E}_h^i} \int_e \left[\!\!\left[\frac{\partial^2(\Pi_h u)}{\partial n_e \partial t_e} \right]\!\!\right] \frac{\partial[(g-u) - E_h\Pi_h(g-u)]}{\partial t_e} ds. \end{aligned} \tag{3.4}$$

The two terms on the right-hand side of (3.4) can be bounded as follows:

$$\begin{aligned} &\left| \sum_{e\in\mathcal{E}_h^i} \int_e \left[\!\!\left[\frac{\partial^2(\Pi_h u)}{\partial n_e^2} \right]\!\!\right] \frac{\partial[(g-u) - E_h\Pi_h(g-u)]}{\partial n_e} ds \right| \\ &\qquad \le \left(\sum_{e\in\mathcal{E}_h^i} |e| \| [\![\partial^2(\Pi_h u)/\partial n_e^2]\!] \|_{L_2(e)}^2 \right)^{\frac{1}{2}} \\ &\qquad\quad \times \left(\sum_{e\in\mathcal{E}_h^i} |e|^{-1} \|\partial[(g-u) - E_h\Pi_h(g-u)]/\partial n_e\|_{L_2(e)}^2 \right)^{\frac{1}{2}} \le Ch^{2\alpha} \end{aligned} \tag{3.5}$$

by Lemma 2.5 and (2.28), and

$$
\begin{aligned}
&\left|\sum_{e\in\mathcal{E}_h^i}\int_e \left[\!\!\left[\frac{\partial^2(\Pi_h u)}{\partial n_e \partial t_e}\right]\!\!\right] \frac{\partial[(g-u)-E_h\Pi_h(g-u)]}{\partial t_e}ds\right| \\
&\qquad \le \left(\sum_{e\in\mathcal{E}_h^i}|e|\|[\![\partial^2(\Pi_h u)/\partial n_e\partial t_e]\!]\|_{L_2(e)}^2\right)^{\frac{1}{2}} \qquad (3.6)\\
&\qquad\qquad \times\left(\sum_{e\in\mathcal{E}_h^i}|e|^{-1}\|\partial[(g-u)-E_h\Pi_h(g-u)]/\partial t_e\|_{L_2(e)}^2\right)^{\frac{1}{2}} \\
&\qquad \le Ch^\alpha\left(\sum_{e\in\mathcal{E}_h^i}|e|^{-1}\|[\![\partial(\Pi_h u)/\partial n_e]\!]\|_{L_2(e)}^2\right)^{\frac{1}{2}} \le Ch^{2\alpha}
\end{aligned}
$$

by (2.13), (2.29), and a standard inverse estimate.

The lemma follows from (3.2)–(3.6). ∎

Lemma 3.2. *We have*

$$
\begin{aligned}
&\left|\sum_{T\in\mathcal{T}_h}\int_T D^2(\Pi_h u):D^2(v-E_hv)]dx+\sum_{e\in\mathcal{E}_h}\int_e\{\!\!\{\partial^2(\Pi_h u)/\partial n^2\}\!\!\}[\![\partial v/\partial n]\!]ds\right| \\
&\qquad \le Ch^\alpha\|v\|_h \qquad \forall\, v\in V_h\cap H_0^1(\Omega).
\end{aligned}
$$

Proof. Let $v\in V_h\cap H_0^1(\Omega)$ be arbitrary. It follows from (2.1) and (2.20) that

$$
\begin{aligned}
&\sum_{T\in\mathcal{T}_h}\int_T D^2(\Pi_h u):D^2(v-E_hv)]dx+\sum_{e\in\mathcal{E}_h}\int_e\{\!\!\{\partial^2(\Pi_h u)/\partial n^2\}\!\!\}[\![\partial v/\partial n]\!]ds \\
&\qquad = -\sum_{e\in\mathcal{E}_h^i}\int_e\left[\!\!\left[\frac{\partial^2(\Pi_h u)}{\partial n_e^2}\right]\!\!\right]\left\{\!\!\left\{\frac{\partial(v-E_hv)}{\partial n_e}\right\}\!\!\right\}ds \\
&\qquad\quad -\sum_{e\in\mathcal{E}_h^i}\int_e\left[\!\!\left[\frac{\partial^2(\Pi_h u)}{\partial n_e\partial t_e}\right]\!\!\right]\frac{\partial(v-E_hv)}{\partial t_e}ds.
\end{aligned}
$$

Moreover, we have, by Lemma 2.5 and (2.25),

$$
\begin{aligned}
&\left|\sum_{e\in\mathcal{E}_h^i}\int_e[\![\partial^2(\Pi_h u)/\partial n_e^2]\!]\{\!\!\{\partial(v-E_hv)/\partial n_e\}\!\!\}ds\right| \\
&\qquad \le \left(\sum_{e\in\mathcal{E}_h^i}|e|\|[\![\partial^2(\Pi_h u)/\partial n_e^2]\!]\|_{L_2(e)}^2\right)^{\frac{1}{2}}\left(\sum_{e\in\mathcal{E}_h^i}|e|^{-1}\|\{\!\!\{\partial(v-E_hv)/\partial n_e\}\!\!\}\|_{L_2(e)}^2\right)^{\frac{1}{2}} \\
&\qquad \le Ch^\alpha\|v\|_h,
\end{aligned}
$$

and

$$\left|\sum_{e\in\mathcal{E}_h^i}\int_e [\![\partial^2(\Pi_h u)/\partial n_e\partial t_e]\!]\frac{\partial(v-E_h v)}{\partial t_e}ds\right|$$
$$\leq \left(\sum_{e\in\mathcal{E}_h^i}|e|\|[\![\partial^2(\Pi_h u)/\partial n_e\partial t_e]\!]\|^2_{L_2(e)}\right)^{\frac{1}{2}}\left(\sum_{e\in\mathcal{E}_h^i}|e|^{-1}\|\partial(v-E_h v)/\partial t_e\|^2_{L_2(e)}\right)^{\frac{1}{2}}$$
$$\leq C\left(\sum_{e\in\mathcal{E}_h^i}|e|^{-1}\|[\![\partial(\Pi_h u)/\partial n]\!]\|^2_{L_2(e)}\right)^{\frac{1}{2}}\|v\|_h \leq Ch^\alpha\|v\|_h$$

by (2.13), (2.26), and a standard inverse estimate. ▯

We are now ready to bound the right-hand side of (3.1) using the variational inequalities (1.6) and (2.16).

LEMMA 3.3. *There exists a positive constant C independent of h such that*

$$a_h(\Pi_h u, \Pi_h u - u_h) - F(\Pi_h u - u_h) \leq C(h^{2\alpha} + h^\alpha\|\Pi_h u - u_h\|_h). \tag{3.7}$$

Proof. We will streamline the arguments using the notation $A \underset{\cdot}{\leq} B$ defined by

$$A \underset{\cdot}{\leq} B \quad \text{if and only if} \quad A - B \leq C(h^{2\alpha} + h^\alpha\|\Pi_h u - u_h\|_h),$$

and rewrite (3.7) as

$$a_h(\Pi_h u, \Pi_h u - u_h) - F(\Pi_h u - u_h) \underset{\cdot}{\leq} 0. \tag{3.8}$$

It follows from (2.6) and (2.7) that

$$\begin{aligned}
&a_h(\Pi_h u, \Pi_h u - u_h) - F(\Pi_h u - u_h)\\
&= \sum_{T\in\mathcal{T}_h}\int_T D^2(\Pi_h u) : D^2(\Pi_h u - u_h)dx\\
&\quad + \sum_{e\in\mathcal{E}_h}\int_e \left\{\!\!\left\{\frac{\partial^2(\Pi_h u)}{\partial n^2}\right\}\!\!\right\}\left[\!\!\left[\frac{\partial(\Pi_h u - u_h)}{\partial n}\right]\!\!\right]ds\\
&\quad + \sum_{e\in\mathcal{E}_h}\int_e \left(\left\{\!\!\left\{\frac{\partial^2(\Pi_h u - u_h)}{\partial n^2}\right\}\!\!\right\} + \frac{\sigma}{|e|}\left[\!\!\left[\frac{\partial(\Pi_h u - u_h)}{\partial n}\right]\!\!\right]\right)\left[\!\!\left[\frac{\partial(\Pi_h u - u)}{\partial n}\right]\!\!\right]ds\\
&\quad - (f, \Pi_h u - u_h),
\end{aligned} \tag{3.9}$$

where we have also used the fact that $[\![\partial u/\partial n]\!] = 0$ on interior edges.

From (2.8) and (2.12) we have

$$\begin{aligned}
&\left|\sum_{e\in\mathcal{E}_h}\int_e \left(\left\{\!\!\left\{\frac{\partial^2(\Pi_h u - u_h)}{\partial n^2}\right\}\!\!\right\} + \frac{\sigma}{|e|}\left[\!\!\left[\frac{\partial(\Pi_h u - u_h)}{\partial n}\right]\!\!\right]\right)\left[\!\!\left[\frac{\partial(\Pi_h u - u)}{\partial n}\right]\!\!\right]ds\right|\\
&\leq C\left[\sum_{e\in\mathcal{E}_h}\left(|e|\left\|\left\{\!\!\left\{\frac{\partial^2(\Pi_h u - u_h)}{\partial n^2}\right\}\!\!\right\}\right\|^2_{L_2(e)} + \frac{1}{|e|}\left\|\left[\!\!\left[\frac{\partial(\Pi_h u - u_h)}{\partial n}\right]\!\!\right]\right\|^2_{L_2(e)}\right)\right]^{\frac{1}{2}}\\
&\qquad \times\left(\sum_{e\in\mathcal{E}_h}\frac{1}{|e|}\|[\![\partial(u - \Pi_h u)/\partial n]\!]\|^2_{L_2(e)}\right)^{\frac{1}{2}}\\
&\leq Ch^\alpha\|\Pi_h u - u_h\|_h.
\end{aligned} \tag{3.10}$$

The sum of the first two terms on the right-hand side of (3.9) can be rewritten as

$$\begin{aligned}
&\sum_{T\in\mathcal{T}_h}\int_T D^2(\Pi_h u):D^2(\Pi_h u-u_h)dx\\
&\qquad+\sum_{e\in\mathcal{E}_h}\int_e \{\!\{\partial^2(\Pi_h u)/\partial n^2\}\!\}[\![\partial(\Pi_h u-u_h)/\partial n]\!]ds\\
&\qquad\qquad=\sum_{T\in\mathcal{T}_h}\int_T D^2(\Pi_h u):D^2E_h(\Pi_h u-u_h)dx\\
&\qquad\qquad\quad+\sum_{T\in\mathcal{T}_h}\int_T D^2(\Pi_h u):D^2[(\Pi_h u-u_h)-E_h(\Pi_h u-u_h)]dx\\
&\qquad\qquad\quad+\sum_{e\in\mathcal{E}_h}\int_e \{\!\{\partial^2(\Pi_h u)/\partial n^2\}\!\}[\![\partial(\Pi_h u-u_h)/\partial n]\!]ds,
\end{aligned}\tag{3.11}$$

and it follows from Lemma 3.2 that

$$\begin{aligned}
&\Bigg|\sum_{T\in\mathcal{T}_h}\int_T D^2(\Pi_h u):D^2[(\Pi_h u-u_h)-E_h(\Pi_h u-u_h)]dx\\
&\quad+\sum_{e\in\mathcal{E}_h}\int_e \{\!\{\partial^2(\Pi_h u)/\partial n^2\}\!\}[\![\partial(\Pi_h u-u_h)/\partial n]\!]ds\Bigg|\le Ch^\alpha\|\Pi_h u-u_h\|_h.
\end{aligned}\tag{3.12}$$

Combining (3.9)–(3.12), we have

$$\begin{aligned}
&a_h(\Pi_h u,\Pi_h u-u_h)-F(\Pi_h u-u_h)\\
&\qquad\lessapprox \sum_{T\in\mathcal{T}_h}\int_T D^2(\Pi_h u):D^2E_h(\Pi_h u-u_h)dx-(f,\Pi_h u-u_h).
\end{aligned}\tag{3.13}$$

We can use (1.5) to rewrite the first term on the right-hand side of (3.13) as

$$\begin{aligned}
&\sum_{T\in\mathcal{T}_h}\int_T D^2(\Pi_h u):D^2E_h(\Pi_h u-u_h)dx\\
&=\sum_{T\in\mathcal{T}_h}\int_T D^2u:D^2E_h(\Pi_h u-u_h)dx+\sum_{T\in\mathcal{T}_h}\int_T D^2(\Pi_h u-u):D^2E_h(\Pi_h u-u_h)dx\\
&=a\big(u,E_h(\Pi_h u-u_h)\big)+\sum_{T\in\mathcal{T}_h}\int_T D^2(\Pi_h u-u):D^2E_h(\Pi_h u-u_h)dx
\end{aligned}$$

and observe that

$$\Bigg|\sum_{T\in\mathcal{T}_h}\int_T D^2(\Pi_h u-u):D^2E_h(\Pi_h u-u_h)dx\Bigg|\le Ch^\alpha\|\Pi_h u-u_h\|_h$$

by (2.12) and (2.24). These relations together with (3.13) imply

$$a_h(\Pi_h u,\Pi_h u-u_h)-F(\Pi_h u-u_h)\lessapprox a\big(u,E_h(\Pi_h u-u_h)\big)-(f,\Pi_h u-u_h).\tag{3.14}$$

Next we use (2.34) to write

$$\begin{aligned}
a\big(u,E_h(\Pi_h u-u_h)\big)&=a(u,T_h\Pi_h u-T_hu_h)\\
&=a(\tilde u_h,T_h\Pi_h u-T_hu_h)+a(u-\tilde u_h,T_h\Pi_h u-T_hu_h)
\end{aligned}$$

and note that

$$|a(u-\tilde{u}_h, T_h\Pi_h u - T_h u_h)| \le C|u-\tilde{u}|_{H^2(\Omega)}|T_h\Pi_h u - T_h u_h|_{H^2(\Omega)} \le Ch\|\Pi_h u - u_h\|_h$$

by (2.17) and (2.32). Moreover, from (2.16) and (2.31) we have

$$\begin{aligned} a(\tilde{u}_h, T_h\Pi_h u - T_h u_h) &= a(\tilde{u}_h, \tilde{u}_h - T_h u_h) + a(\tilde{u}_h, T_h\Pi_h u - \tilde{u}_h) \\ &\le (f, \tilde{u}_h - T_h u_h) + a(\tilde{u}_h, T_h\Pi_h u - \tilde{u}_h). \end{aligned}$$

Therefore it follows from (3.14) that

$$\begin{aligned} &a_h(\Pi_h u, \Pi_h u - u_h) - F(\Pi_h u - u_h) \\ &\quad \lessapprox a(\tilde{u}_h, T_h\Pi_h u - \tilde{u}_h) - (f, \Pi_h u - u_h - \tilde{u}_h + T_h u_h). \end{aligned} \tag{3.15}$$

We rewrite the first term on the right-hand side of (3.15) as

$$\begin{aligned} a(\tilde{u}_h, T_h\Pi_h u - \tilde{u}_h) = a(\tilde{u}_h - u, T_h\Pi_h u - \tilde{u}_h) + a(u, T_h\Pi_h u - u) \\ + a(u, u - \tilde{u}_h) \end{aligned}$$

and observe that

$$|a(\tilde{u}_h - u, T_h\Pi_h u - \tilde{u}_h)| \le |\tilde{u}_h - u|_{H^2(\Omega)}\big(|T_h\Pi_h u - u|_{H^2(\Omega)} + |u-\tilde{u}_h|_{H^2(\Omega)}\big) \le Ch^{1+\alpha}$$

by (2.17) and (2.33). Moreover we have, by Lemma 3.1,

$$|a(u, T_h\Pi_h u - u)| \le Ch^{2\alpha}.$$

Hence we obtain from (3.15) the relation

$$a_h(\Pi_h u, \Pi_h u - u_h) - F(\Pi_h u - u_h) \lessapprox a(u, u-\tilde{u}_h) - (f, \Pi_h u - u_h - \tilde{u}_h + T_h u_h). \tag{3.16}$$

Now we use (1.6) and (2.18) to derive

$$\begin{aligned} a(u, u-\tilde{u}_h) &= a(u, u-\hat{u}_h) + \delta_{h,1}a(u,\phi_1) - \delta_{h,2}a(u,\phi_2) \\ &\le (f, u-\hat{u}_h) + \delta_{h,1}a(u,\phi_1) - \delta_{h,2}a(u,\phi_2) \\ &= (f, T_h\Pi_h u - \tilde{u}_h) + (f, u - T_h\Pi_h u) \\ &\qquad - \delta_{h,1}[(f,\phi_1) - a(u,\phi_1)] + \delta_{h,2}[(f,\phi_2) - a(u,\phi_2)], \end{aligned}$$

and we note that, by (2.19) and (2.33),

$$\begin{aligned} &|(f, u - T_h\Pi_h u)| \le \|f\|_{L_2(\Omega)}\|u - T_h\Pi_h u\|_{L_2(\Omega)} \le Ch^{2+\alpha}, \\ &|\delta_{h,1}[(f,\phi_1) - a(u,\phi_1)]| + |\delta_{h,2}[(f,\phi_2) - a(u,\phi_2)]| \le Ch^2. \end{aligned}$$

Therefore we deduce from (3.16) the relation

$$a_h(\Pi_h u, \Pi_h u - u_h) - F(\Pi_h u - u_h) \lessapprox -\big(f, (\Pi_h u - u_h) - (T_h\Pi_h u - T_h u_h)\big). \tag{3.17}$$

Finally, we have by (2.24)

$$\begin{aligned} \big|\big(f, (\Pi_h u - u_h) - (T_h\Pi_h u - T_h u_h)\big)\big| &= \big|\big(f, \Pi_h u - u_h - E_h(\Pi_h u - u_h)\big)\big| \\ &\le \|f\|_{L_2(\Omega)}\|\Pi_h u - u_h - E_h(\Pi_h u - u_h)\|_{L_2(\Omega)} \le Ch^2\|\Pi_h u - u_h\|_h, \end{aligned}$$

which together with (3.17) yields (3.8). ∎

THEOREM 3.4. *There exists a positive constant C independent of h such that*

$$\|u - u_h\|_h \leq Ch^\alpha.$$

Proof. It follows from (3.1), (3.7), and the inequality of arithmetic and geometric means that

$$\|\Pi_h u - u_h\|_h^2 \leq C(h^{2\alpha} + h^\alpha \|\Pi_h u - u_h\|_h) \leq Ch^{2\alpha} + \frac{1}{2}\|\Pi_h u - u_h\|_h^2,$$

which implies

$$\|\Pi_h u - u_h\|_h \leq Ch^\alpha. \tag{3.18}$$

The theorem follows from (2.12), (3.18), and the triangle inequality. ∎

4. Approximations of the coincidence set and the free boundary. In this section we consider the approximations of the coincidence set (resp., free boundary) by the discrete coincidence sets (resp., discrete free boundaries). We begin with an error estimate in the L_∞ norm.

THEOREM 4.1. *There exists a positive constant C independent of h such that*

$$\|u - u_h\|_{L_\infty(\Omega)} \leq Ch^\alpha. \tag{4.1}$$

Proof. We have

$$\begin{aligned} \|u - u_h\|_{L_\infty(\Omega)} \leq \|u - \Pi_h u\|_{L_\infty(\Omega)} &+ \|(\Pi_h u - u_h) - E_h(\Pi_h u - u_h)\|_{L_\infty(\Omega)} \\ &+ \|E_h(\Pi_h u - u_h)\|_{L_\infty(\Omega)}, \end{aligned} \tag{4.2}$$

$$\|u - \Pi_h u\|_{L_\infty(\Omega)} \leq Ch^{1+\alpha}|u|_{H^{2+\alpha}(\Omega)}, \tag{4.3}$$

$$\begin{aligned} &\|(\Pi_h u - u_h) - E_h(\Pi_h u - u_h)\|_{L_\infty(\Omega)} \\ &\quad = \max_{T\in\mathcal{T}_h} \|(\Pi_h u - u_h) - E_h(\Pi_h u - u_h)\|_{L_\infty(T)} \\ &\quad \leq C \max_{T\in\mathcal{T}_h} h_T^{-1}\|(\Pi_h u - u_h) - E_h(\Pi_h u - u_h)\|_{L_2(T)} \leq Ch\|\Pi_h u - u_h\|_h \end{aligned} \tag{4.4}$$

by standard interpolation and inverse estimates (cf. [23, 13]) and (2.22).

We can estimate the last term on the right-hand side of (4.2) by the Sobolev inequality [1], a Poincaré–Friedrichs inequality [41], and (2.24):

$$\begin{aligned} \|E_h(\Pi_h u - u_h)\|_{L_\infty(\Omega)} &\leq C\|E_h(\Pi_h u - u_h)\|_{H^2(\Omega)} \\ &\leq C|E_h(\Pi_h u - u_h)|_{H^2(\Omega)} \leq C\|\Pi_h u - u_h\|_h. \end{aligned} \tag{4.5}$$

The estimate (4.1) follows from (3.18) and (4.2)–(4.5). ∎

Remark 4.2. Numerical results in section 5 indicate that the estimate (4.1) is not sharp.

Let I_i $(i = 1, 2)$ be the coincidence sets of the obstacle problem (1.2), i.e., $I_i = \{x \in \Omega : u(x) = \psi_i(x)\}$, and let $F_i = \partial I_i$ $(i = 1, 2)$ be the corresponding free boundaries.

The discrete coincidence sets $I_{h,i}$ $(i = 1, 2)$ are defined by

$$I_{h,1} = \{x \in \Omega : u_h(x) - \psi_1(x) \leq \tau_h\}, \tag{4.6}$$

$$I_{h,2} = \{x \in \Omega : \psi_2(x) - u_h(x) \leq \tau_h\}, \tag{4.7}$$

$$\tau_h = \rho\|u - u_h\|_{L_\infty(\Omega)}, \tag{4.8}$$

and ρ can be any number > 1. Observe that (4.6)–(4.8) imply, for $i = 1, 2$,

$$I_i \subseteq I_{h,i} \quad \text{and} \quad I_{h,i} \setminus I_i \subseteq \{x \in \Omega : 0 < |u(x) - \psi_i(x)| \leq 2\tau_h\}. \tag{4.9}$$

It follows from (1.1), (4.1), and (4.9) that the discrete coincidence sets are disjoint compact subsets of Ω if h is sufficiently small, which is assumed to be the case. We then define $F_{h,i} = \partial I_{h,i}$ ($i = 1, 2$) to be the discrete free boundaries.

We can obtain an approximation result for the coincidence set under the following nondegeneracy assumption (cf. [19, 42]): There exist positive numbers μ_1 and μ_2 such that

$$\big|\{x \in \Omega : 0 < |u(x) - \psi_i(x)| \leq \epsilon\}\big| \leq C\epsilon^{\mu_i} \quad (i = 1, 2) \tag{4.10}$$

for $\epsilon > 0$ sufficiently small, where $|A|$ denotes the Lebesgue measure of a set A and the positive constant C is independent of ϵ. Indeed (4.9) and (4.10) immediately imply the following result on the symmetric difference $I_i \Delta I_{h,i}$ of I_i and $I_{h,i}$:

$$|I_i \Delta I_{h,i}| \leq C\tau_h^{\mu_i} \quad \text{for} \quad i = 1, 2. \tag{4.11}$$

The approximation result for the free boundary requires the following stronger nondegeneracy assumption (cf. [19, 42]): There exist positive numbers μ_1 and μ_2 such that

$$\{x \in \Omega : 0 < |u(x) - \psi_i(x)| \leq \epsilon\} \subseteq \{x \in \Omega : \operatorname{dist}(x, F_i) \leq C\epsilon^{\mu_i}\} \quad (i = 1, 2) \tag{4.12}$$

for $\epsilon > 0$ sufficiently small, where the positive constant C is independent of ϵ.

Observe that (4.6)–(4.8) imply

$$F_{h,i} \subseteq \{x \in \Omega : 0 < |u(x) - \psi_i(x)| \leq 2\tau_h\} \tag{4.13}$$

since $|u_h(x) - \psi_i(x)| = \tau_h$ for any $x \in F_{h,i}$ and $\tau_h > \|u - u_h\|_\infty$ (except in the trivial case where $u = u_h$). It then follows from (4.12) and (4.13) that

$$F_{h,i} \subseteq \{x \in \Omega : \operatorname{dist}(x, F_i) \leq C\tau_h^{\mu_i}\}; \tag{4.14}$$

i.e., the discrete free boundary $F_{h,i}$ is within a tubular neighborhood of the continuous free boundary F_i whose width is $O(\tau_h^{\mu_i})$.

Remark 4.3. For second order elliptic obstacle problems, it is possible to establish (4.10) and (4.12) under appropriate assumptions on the obstacle functions and the load function (cf. [21, 42]). But such results are not available for plates. Indeed there are only limited theoretical results concerning the free boundaries of fourth order obstacle problems [22, 43]. Therefore (4.10) and (4.12) are assumptions that need to be verified for individual plate obstacle problems.

5. Numerical results.

In this section we present numerical results for several one-obstacle problems to demonstrate the performance of the quadratic C^0 interior penalty method. The obstacle function from below will be denoted by ψ.

First we give a brief description of the algorithm that we use to solve the discrete obstacle problem. Let $n_h = \dim V_h$ and $\{\varphi_i\}_{i=1}^{n_h}$ be the nodal basis of V_h. We can express the discrete optimization problem (2.5) as the following box constrained convex quadratic programming problem: Find $\mathbf{u} \in \mathbb{R}^{n_h}$ such that

$$\mathbf{u} = \operatorname*{argmin}_{\mathbf{v} \in \mathcal{B}} \mathcal{Q}(\mathbf{v}), \tag{5.1}$$

where $\mathcal{B} = \{\mathbf{v} \in \mathbb{R}^{n_h} : \psi(p_i) \leq \mathbf{v}_i \quad \forall p_i \in \mathcal{V}_h \cap \Omega, \mathbf{v}_i = g(p_i), \quad \forall p_i \in \mathcal{V}_h \cap \partial\Omega\}$, $\mathcal{Q}(\mathbf{v}) = \frac{1}{2}\mathbf{v}^T\mathbf{A}_h\mathbf{v} - \mathbf{b}^T\mathbf{v}$, $\mathbf{b}_i = F(\varphi_i)$, and $\mathbf{A}_h$ is the stiffness matrix whose (i,j) component is $a_h(\varphi_j, \varphi_i)$.

We solve (5.1) by an active set algorithm developed in [35]. It consists of a nonmonotone gradient projection step that uses the cyclic Barzilai–Borwein algorithm [24], an unconstrained optimization step that uses the conjugate gradient algorithm CG_DESCENT [34], and a set of rules for branching between these two steps that either restart the nonmonotone gradient projection step or restart the unconstrained optimization step. It is shown in [33] that this active set algorithm eventually reduces (5.1) to an unconstrained strongly convex quadratic problem, which implies that theoretically (5.1) can be solved in a finite number of iterations. Numerical experiments in [33] show that this algorithm performs very well in comparison with other well-established methods for box constrained optimization.

We use the stopping criterion $\|\mathcal{P}[\mathbf{v} - (\nabla\mathcal{Q})(\mathbf{v})] - \mathbf{v}\|_\infty \leq 10^{-8}$ for the examples in this paper, where $\mathcal{P}$ is the projection onto the set $\mathcal{B}$, and we have the following result [35] regarding the distance between $\mathbf{v}$ and $\mathbf{u}$:

$$\|\mathbf{v} - \mathbf{u}\|_2 \leq C\|\mathcal{P}[\mathbf{v} - (\nabla\mathcal{Q})(\mathbf{v})] - \mathbf{v}\|_2,$$

where $\|\cdot\|_2$ denotes the Euclidean norm and the positive constant C depends on the condition number of $\mathbf{A}_h$.

We take the penalty parameter σ to be 5 in the following examples.

Example 1. In this example we apply the quadratic C^0 interior penalty method to a problem with a known exact solution to validate the numerical results. We begin with the plate obstacle problem on the disc $\{x : |x| < 2\}$ with $f = 0$, $\psi(x) = 1-|x|^2$, and homogeneous Dirichlet boundary conditions. This problem is rotationally invariant and can be solved exactly. The exact solution is given by

$$u(x) = \begin{cases} C_1|x|^2(\ln|x|) + C_2|x|^2 + C_3(\ln|x|) + C_4, & r_0 < |x| < 2, \\ 1 - |x|^2, & |x| \leqslant r_0, \end{cases} \tag{5.2}$$

where $r_0 \approx 0.18134452$, $C_1 \approx 0.52504063$, $C_2 \approx -0.62860904$, $C_3 \approx 0.01726640$, and $C_4 \approx 1.04674630$.

We then consider the obstacle problem on $\Omega = (-0.5, 0.5)^2$ whose exact solution is the restriction of u to Ω. For this problem $f = 0$, $\psi(x) = 1 - |x|^2$, and the (nonhomogeneous) Dirichlet boundary data are determined by u.

We solve the discrete obstacle problems on uniform triangulations of Ω, where the length of the horizontal/vertical edges of the triangles in the jth level is $h_j = 2^{-j}$, and we denote the energy norm on the jth level by $\|\cdot\|_j$.

Let u_j be the numerical solution of the jth level discrete obstacle problem and let $e_j = \Pi_j u - u_j$, where Π_j is the interpolation operator for the jth level P_2 Lagrange finite element space. We evaluate the error $\|e_j\|_j$ in the energy norm and the error $\|e_j\|_\infty = \max_{p\in\mathcal{N}_j} |e_j(p)|$, where $\mathcal{N}_j$ is the set of the nodal points (vertices and midpoints) in the jth level triangulation. We also compute the rates of convergence in these norms by

$$\beta_h = \ln(\|e_{j-1}\|_{j-1}/\|e_j\|_j)/\ln 2 \quad \text{and} \quad \beta_\infty = \ln(\|e_{j-1}\|_\infty/\|e_j\|_\infty)/\ln 2.$$

The numerical results are presented in Table 5.1. It is observed that the magnitude of the energy (resp., ℓ_∞) norm error is $O(h^{1.5})$ (resp., $O(h^2)$), which is better than the error estimate in Theorem 3.4 (resp., Theorem 4.1). We believe this is likely due

TABLE 5.1
Energy norm errors and ℓ_∞ norm errors for Example 1.

j	$\|e_j\|_j/\|u_8\|_8$	β_h	$\|e_j\|_\infty$	β_∞
1	3.4440×10^{-2}		1.0761×10^{-2}	
2	1.8146×10^{-2}	0.9245	3.5160×10^{-3}	1.6137
3	6.1763×10^{-3}	1.5548	6.2684×10^{-4}	2.4878
4	2.1912×10^{-3}	1.4950	1.4770×10^{-4}	2.0855
5	9.2498×10^{-4}	1.2442	7.5174×10^{-5}	0.9743
6	3.6448×10^{-4}	1.3436	2.6261×10^{-5}	1.5173
7	1.2529×10^{-4}	1.5405	6.7526×10^{-6}	1.9594
8	4.6397×10^{-5}	1.4332	1.7058×10^{-6}	1.9850

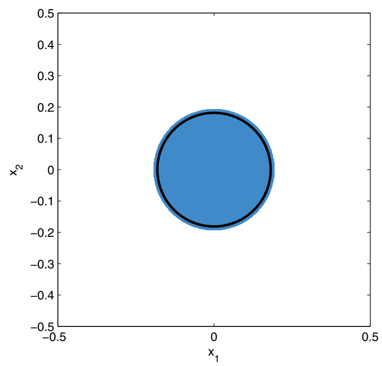


FIG. 5.1. *Discrete coincidence set and exact free boundary for Example* 1.

to the effects of superconvergence because the solution u is a piecewise C^∞ function and we use uniform grids in the computation.

We also consider the approximations of the coincidence set and the free boundary. From (5.2) we see that the continuous coincidence set is $I = \{x \in \Omega : |x| \le r_0\}$, and the continuous free boundary is $F = \{x \in \Omega : |x| = r_0\}$. A simple calculation using Taylor's theorem shows that the assumptions (4.10) and (4.12) are valid for $\mu = 1/3$.

For simplicity, we take the jth level discrete coincidence set to be

$$I_j = \{p \in \mathcal{N}_j : u_j(p) - \psi(p) \le \|e_j\|_\infty\}. \tag{5.3}$$

The discrete coincidence set in level 8 is the disc displayed in Figure 5.1, where the circle represents the exact free boundary F.

We compute the convergence rates for the coincidence set and the free boundary by

$$\beta_c = \ln(m_{j-1}/m_j)/\ln 2 \quad \text{and} \quad \beta_b = \ln(d_{j-1}/d_j)/\ln 2,$$

where m_j is the Lebesgue measure of $I \Delta I_j$ and $d_j = \operatorname{dist}(F_j, F)$, and tabulate the results in Table 5.2. According to (4.11) and (4.14), the magnitude of the errors should be $O(\|e_j\|_\infty^\mu) = O(h^{2/3})$, which is in agreement with the numerical results.

Example 2. In this example we take $\Omega = (-0.5, 0.5)^2$, $f = g = 0$, and $\psi(x) = 1 - 5|x|^2 + |x|^4$. We solve the discrete obstacle problems on the same uniform triangulations as in Example 1.

TABLE 5.2
Approximations of the coincidence set and the free boundary for Example 1.

j	m_j	β_c	d_j	β_b
4	1.7895×10^{-2}		4.4002×10^{-2}	
5	1.4233×10^{-2}	0.3303	3.5725×10^{-2}	0.3007
6	9.3505×10^{-3}	0.6062	2.4170×10^{-2}	0.5637
7	5.8105×10^{-3}	0.6864	1.5369×10^{-2}	0.6532
8	3.5674×10^{-3}	0.7038	9.5828×10^{-3}	0.6815

Since the exact solution is not known, we take $\tilde{e}_j = u_{j-1} - u_j$ and compute the rates of convergence $\tilde{\beta}_h$ and $\tilde{\beta}_\infty$ by

$$\tilde{\beta}_h = \ln(\|\tilde{e}_{j-1}\|_{j-1}/\|\tilde{e}_j\|_j)/\ln 2 \quad \text{and} \quad \tilde{\beta}_\infty = \ln(\|\tilde{e}_{j-1}\|_\infty/\|\tilde{e}_j\|_\infty)/\ln 2.$$

The numerical results are presented in Table 5.3. It is observed that the magnitude of the energy norm error is $O(h)$, as predicted by Theorem 3.4. The results for the ℓ_∞ norm errors suggest that the correct estimate for $\|u - u_h\|_{L_\infty(\Omega)}$ is $O(h^2)$ for this example.

TABLE 5.3
Energy norm errors and ℓ_∞ norm errors for Example 2.

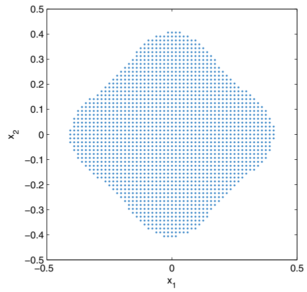
j	$\|\tilde{e}_j\|_j/\|u_8\|_8$	$\tilde{\beta}_h$	$\|\tilde{e}_j\|_\infty$	$\tilde{\beta}_\infty$
1	3.2401×10^{-1}		1.0000×10^{0}	
2	4.5394×10^{-1}	−0.4865	3.4417×10^{-1}	1.5388
3	4.9944×10^{-1}	−0.1378	5.9705×10^{-2}	2.5272
4	3.8333×10^{-1}	0.3817	2.6127×10^{-2}	1.1923
5	1.9609×10^{-1}	0.9670	3.6557×10^{-3}	2.8373
6	9.2707×10^{-2}	1.0808	1.2895×10^{-3}	1.5033
7	4.4712×10^{-2}	1.0520	4.1668×10^{-4}	1.6298
8	2.1855×10^{-2}	1.0327	1.0245×10^{-4}	2.0240

The discrete coincidence sets for levels 5–8 are displayed in Figure 5.2, where we replace e_j with $\tilde{e}_j$ in (5.3). It is observed that I_j converges to a domain with a smooth boundary and the correct symmetry. Since $\Delta^2\psi > 0$ in this example, the noncoincidence set $\Omega \setminus I$ is connected (cf. [22, section 8]). This is confirmed by Figure 5.2.

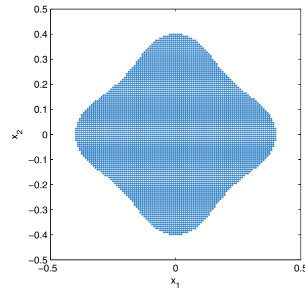
Example 3. In this example we take $\Omega = (-0.5, 0.5)^2$, $f = g = 0$, and $\psi(x) = 1 - 5|x|^2 - |x|^4$. We solve the discrete obstacle problems on the same uniform triangulations as in Example 1. The numerical results for $\|\tilde{e}_j\|_j$, $\|\tilde{e}_j\|_\infty$, $\tilde{\beta}_h$, and $\tilde{\beta}_\infty$ are presented in Table 5.4. It is observed that the magnitude of the energy norm error is $O(h)$. The results for the ℓ_∞ norm errors suggest that the correct estimate for $\|u - u_h\|_{L_\infty(\Omega)}$ is $O(h^2)$ for this example.

Note that this example is similar to Example 2 except in the sign of the term $|x|^4$ that appears in the obstacle function. Here we have $\Delta^2\psi < 0$ in Ω, and hence the interior of the coincidence set must be empty since $\Delta^2 u$ (in the sense of distributions) is a nonnegative measure (cf. [22, section 8]). This is confirmed by the pictures of the discrete coincidence sets I_7 and I_8 displayed in Figure 5.3. Moreover, the free boundary appears to be smooth.

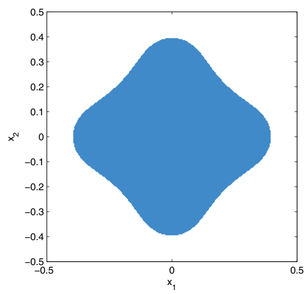
Example 4. In this example we take Ω to be the L-shaped domain $(-0.5, 0.5)^2 \setminus [0, 0.5]^2$, $f = g = 0$, and $\psi(x) = 1 - \big[\frac{(x_1+0.25)^2}{0.2^2} + \frac{x_2^2}{0.35^2}\big]$.



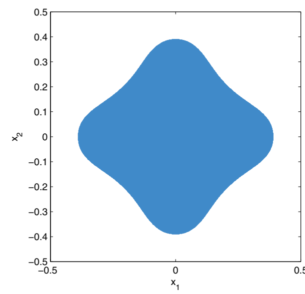
(a) I_5



(b) I_6



(c) I_7



(d) I_8

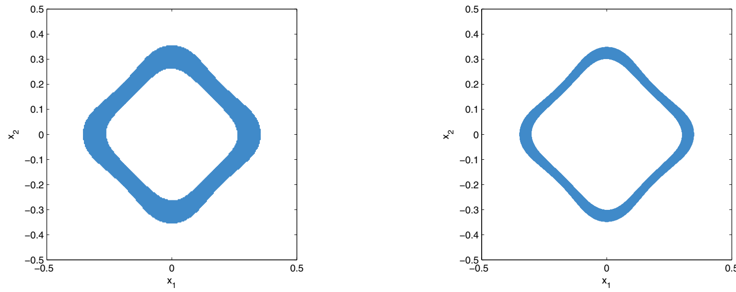
FIG. 5.2. *Discrete coincidence sets for Example* 2.

TABLE 5.4
Energy norm errors and ℓ_∞ norm errors for Example 3.

j	$\|\tilde{e}_j\|_j/\|u_8\|_8$	$\tilde{\beta}_h$	$\|\tilde{e}_j\|_\infty$	$\tilde{\beta}_\infty$
1	3.4133×10^{-1}		1.0000×10^{0}	
2	4.7596×10^{-1}	-0.4797	3.3309×10^{-1}	1.5860
3	5.1117×10^{-1}	-0.1030	7.2578×10^{-2}	2.1983
4	3.3897×10^{-1}	0.5926	2.5308×10^{-2}	1.5199
5	1.6913×10^{-1}	1.0030	7.6540×10^{-3}	1.7253
6	7.9146×10^{-2}	1.0956	1.6226×10^{-3}	2.2380
7	3.8567×10^{-2}	1.0371	5.8201×10^{-4}	1.4791
8	1.8889×10^{-2}	1.0298	1.0995×10^{-4}	2.4042

We solve the discrete obstacle problems on uniform triangulations, where the jth level mesh size is $h_j = 2^{-(j+1)}$. The energy norm errors and ℓ_∞ norm errors are presented in Table 5.5. The magnitude of the observed energy norm error is consistent with Theorem 3.4, since the index of elliptic regularity α is less than 1 for the L-shaped domain. In fact we have $\alpha \approx 0.5445$, and the energy norm error at level 7 has not reached the asymptotic region. The results for the ℓ_∞ norm errors strongly suggest that the correct estimate for $\|u - u_h\|_{L_\infty(\Omega)}$ is $O(h^{1+\alpha})$ for this example.

The discrete coincidence sets for levels 6–7 are depicted in Figure 5.4. It is observed that I_j converges to a domain with a smooth boundary. The noncoincidence set is connected, which agrees with the result in [22, section 8] since $\Delta^2\psi = 0$ in Ω in this example.

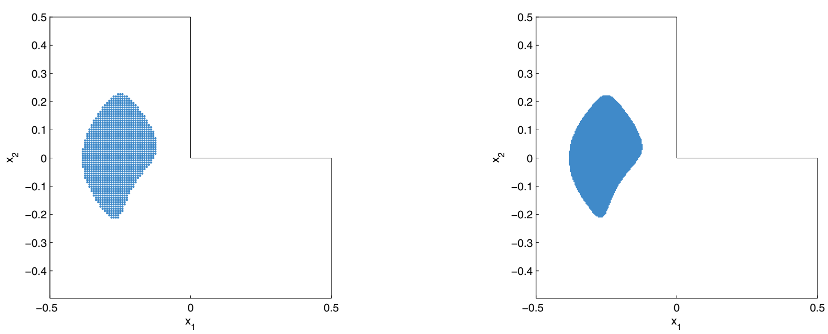


(a) I_7 (b) I_8

FIG. 5.3. *Discrete coincidence sets for Example* 3.

TABLE 5.5
Energy norm errors and ℓ_∞ norm errors for Example 4.

j	$\|\tilde{e}_j\|_j/\|u_7\|_7$	$\tilde{\beta}_h$	$\|\tilde{e}_j\|_\infty$	$\tilde{\beta}_\infty$
1	3.8757×10^{-1}		1.0000×10^{0}	
2	5.7107×10^{-1}	−0.5592	2.1135×10^{-1}	2.2423
3	4.4676×10^{-1}	0.3542	4.5224×10^{-2}	2.2245
4	2.3225×10^{-1}	0.9438	1.4043×10^{-2}	1.6872
5	1.1700×10^{-1}	0.9893	5.4277×10^{-3}	1.3715
6	6.2281×10^{-2}	0.9095	1.7170×10^{-3}	1.6605
7	3.5177×10^{-2}	0.8241	5.8861×10^{-4}	1.5445



(a) I_6 (b) I_7

FIG. 5.4. *Discrete coincidence sets for Example* 4.

6. Concluding remarks. We have investigated a quadratic C^0 interior penalty method for the obstacle problem of clamped Kirchhoff plates and extended the results in [16] for convex polygons and homogeneous Dirichlet boundary conditions to general polygons and general Dirichlet boundary conditions. By allowing nonhomogeneous boundary conditions we are able to construct an example with a known exact solution to validate the numerical results.

We have proved that the magnitudes of both the energy norm error and the L_∞ norm error are $O(h^\alpha)$, where $\alpha \in (\frac{1}{2}, 1]$ is the index of elliptic regularity determined by the interior angles of the domain. Numerical results indicate that the energy norm error estimate is sharp, but the L_∞ norm error estimate is not. For the examples in section 5 the magnitudes of the L_∞ norm errors appear to be $O(h^{1+\alpha})$.

Since very little is known about the free boundaries of such obstacle problems, we can only derive results for the approximations of the coincidence set and the free boundary by assuming certain nondegeneracy conditions. However, numerical experiments (cf. Figures 5.2–5.4) suggest that it may be possible to obtain regularity results of the free boundaries under appropriate assumptions on the obstacle functions and the load function. This in turn may lead to conditions that are easy to verify and that imply the nondegeneracy conditions.

It is also of interest to develop multilevel solvers for the discrete obstacle problems, and we plan to address this in the near future.

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