Hypergeometric Functions and Modular Forms

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The Explicit Hypergeometric-Modularity Method (EHMM)

by **Michael Allen, Brian Grove, L. and Fang-Ting Tu**

- Part I: **Adv. in Math. 478 (2025), 110411**
- Part II: **Res. in Math. Sci.**(to appear), *arXiv* : 2411.15116

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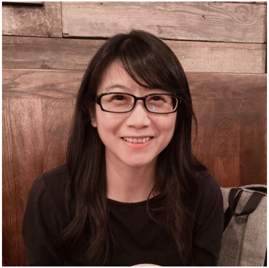


Fang-Ting Tu

Motivation: Riemann zeta function

$$\zeta(s) \quad := \quad \sum_{n \geq 1} \frac{1}{n^s}, \quad \Re(s) > 1.$$

Product form: $\zeta(s) = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}}$

Special values: $\zeta(2n) = \frac{(-1)^{n+1}}{2(2n)!} B_{2n} \cdot (2\pi)^{2n}$, B_n: Bernoulli #s

Meromorphic continuation:

$$\Lambda(s) := \pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s),$$

then
$$\Lambda(s) = \Lambda(1 - s).$$

Modular form connection

$$\theta_3(\tau) := \sum_{n\in\mathbb{Z}} q^{n^2} = \sum_{n\geq 0} a_n q^n = 1 + 2q + 2q^4 + \cdots, \quad q = e^{\pi i \tau}.$$

Mellin transform (Dirichlet Series): $\sum_{n\geq 1} \frac{a_n}{n^{s/2}} = \sum_{n\geq 1} \frac{2}{n^s} = 2\zeta(s).$

Poisson summation: $\theta_3(\sqrt{-1}s) = \frac{1}{\sqrt{s}}\theta_3(\frac{\sqrt{-1}}{s})$, $s \in \mathbb{R}_{>0}$.

$\theta_3(\tau)$ transforms accordingly to $\left\langle \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 4 & 0 \end{pmatrix} \right\rangle$ with a weight $1/2$ factor.

Arithmetic Hypergeometric functions

An **arithmetic hypergeometric datum**, $HD = \{\alpha, \beta\}$, consists of

$$\alpha = \{a_1, ..., a_n\}, \quad \beta = \{b_1 = 1, b_2, ..., b_n\}, \quad a_i, b_j \in \mathbb{Q}.$$

- $n = \#\alpha = \#\beta$, the length of HD
- $M(HD) := lcd(\alpha \cup \beta)$, the least common denom. of the a_i, b_j;
- $$F(\alpha, \beta; t) := {}_nF_{n-1}\left[\begin{matrix} a_1 & a_2 & \cdots & a_n \\ & b_2 & \cdots & b_n \end{matrix}; \ t\right] = \sum_{k \geq 0} C_{HD}(k) \cdot t^k$$
$$C_{HD}(k) := \frac{(a_1)_k \cdots (a_n)_k}{k!(b_2)_k \cdots (b_n)_k},$$

where $(a)_k = a(a+1)\cdots(a+k-1) = \Gamma(a+k)/\Gamma(a)$.

- $F(\alpha, \beta; t)_m = \sum_{k=0}^{m} C_{HD}(k) t^k$

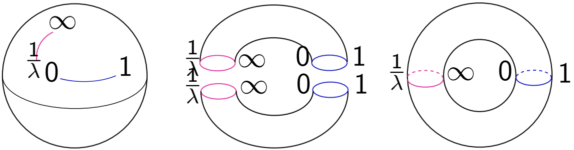
Connection to varieties

Example ($HD_2 = \{\{\frac{1}{2}, \frac{1}{2}\}, \{1, 1\}\}$)

For each $\lambda \in \mathbb{Q} \setminus \{0, 1\}$, the cubic equation

$$L_\lambda : \quad y^2 = x(1-x)(1-\lambda x)$$

represents an elliptic curve, obtained as follows.



$$\Omega_\lambda = \frac{\mathrm{d}x}{y} = \frac{\mathrm{d}x}{\sqrt{x(1-x)(1-\lambda x)}}.$$

$$p(\lambda) := \int_0^1 \Omega_\lambda = \pi \cdot {}_2F_1 \left[\begin{matrix} \frac{1}{2} & \frac{1}{2} \\ & 1 \end{matrix} ; \lambda \right].$$

A version of hypergeometric character sum by McCarthy

Let $\mathbb{F}_p$ be a finite field.

$\widehat{\mathbb{F}_p^\times} := \mathrm{Hom}(\mathbb{F}_p^\times, \mathbb{C}^\times) = \langle \omega \rangle = \{\omega^k \mid 0 \le k \le p-2\}$.

For $p \equiv 1 \mod M$, $\lambda \in \mathbb{F}_p$, let

$$H_p(\alpha, \beta; \lambda) := \sum_{k=0}^{p-2} \mathfrak{C}_{HD}(k) \cdot \omega^k(\lambda)$$

$$\mathfrak{C}_{HD}(k) := \frac{1}{1-p} \prod_{j=1}^{n} \frac{\mathfrak{g}(\omega^{a_j(p-1)+k})}{\mathfrak{g}(\omega^{a_j(p-1)})} \frac{\mathfrak{g}(\omega^{-b_j(p-1)+k})}{\mathfrak{g}(\omega^{-b_j(p-1)})} \cdot \omega^k((-1)^n),$$

where $\mathfrak{g}(\chi) := \sum_{x \in \mathbb{F}_p} \zeta_p^x \cdot \chi(x)$ (Gauss sum of χ).

Example ($HD_2 = \{\{\frac{1}{2}, \frac{1}{2}\}, \{1, 1\}\}$, p odd, $\lambda \in \mathbb{F}_p$)

$$H_p(HD_2; \lambda) = p + 1 - \sum_{x \in \mathbb{F}_p} \left(\frac{x(x-1)(1-\lambda x)}{p} \right) = a_p(\lambda)$$

Euler factors

Using the norm map, one can extend it to $H_{p^k}(\alpha, \beta; \lambda)$ over $\mathbb{F}_{p^k}$.
By Dwork, Katz

$$P(\alpha, \beta; \lambda; p^{-s}) := \exp\left(\sum_{k\geq 1}(-1)^{n-1}H_{p^k}(\alpha, \beta; \lambda)\frac{p^{-sk}}{k}\right)$$

is a **polynomial** of p^{-s}.

Example (HD_2, p odd, $\lambda \in \mathbb{F}_p \setminus \{0, 1\}$)

$$P(HD_2; \lambda; p^{-s}) = \left(1 - a_p(\lambda)p^{-s} + p \cdot p^{-2s}\right).$$

Hypergeometric L-functions

$$L(HD;\lambda;s) \text{“}=\text{''} \prod_p P(HD;\lambda;p^{-s}).$$

Example $(HD_2,\ \lambda \in \mathbb{Q}\setminus\{0,1\})$

$$\begin{aligned} L(HD_2;\lambda;s) \quad &\text{“}=\text{''} \quad \prod \left(1 - a_p(\lambda)p^{-s} + p\cdot p^{-2s}\right)^{-1} \\ &= \quad L(f_{L_\lambda}, s), \end{aligned}$$

where f_{L_λ} is a weight-2 modular form depending on λ.

Hypergeometric L-functions

Photo credit: Amazon

Varying *HD* and λ leds to a database of hypergeometric L-functions, see L-functions and Modular Forms Database (LMFDB) (beta version), (Roberts, Rodriguez Villegas, Watkins, Kedlaya, Roe, ...)

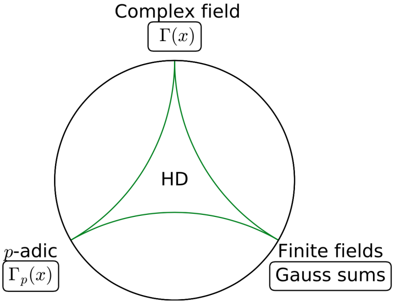
Analytic/meromorphic continuation of $L(HD; \lambda; s)$?

Example ($HD = \{\{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\}, \{1, 1, 1, 1\}\}; \lambda = 1$)

a). **Yes!** $L(HD; 1; s) = L(f_{8.4.a.a}, s) \cdot \zeta(1 + s)$ [1]
[Ahlgren-Ono] For odd prime p, $H_p(HD; 1) = a_p(f_{8.4.a.a}) + p$

b). [Kilbourn] ${}_4F_3(HD; 1)_{p-1} \equiv a_p(f_{8.4.a.a}) \mod p^3$

c). [Zagier] $L(f_{8.4.a.a}, 2) = \frac{\pi^2}{16}\, {}_4F_3(HD; 1)$



[1] $f_{8.4.a.a}$ is in LMFDB label; $a_p(f)$ is the p^{th} coeff. of f.

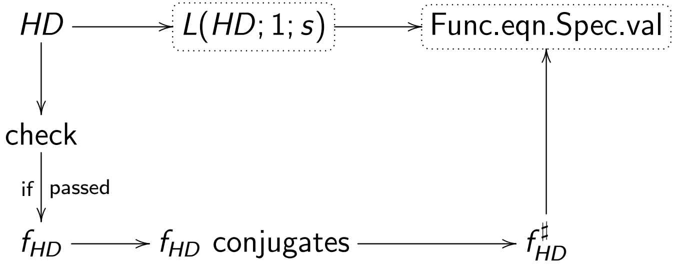
Dawsey–McCarthy conjectures

	Hyp Series	Newform $f(z)=\sum a(n)q^n$		Connection	
	Parameters	**Space**	**LMFDB**	**Relationship**	**Conditions**
1.	$[\frac{1}{3},\frac{1}{2},\frac{1}{2};1,1,1\|1]$	$S_3(\Gamma_0(48),(\frac{-4}{\cdot}))$	48.3.g.a	$a(p)=S_6(p)\cdot F(\cdots)_p$	$p\equiv 1 \pmod 6$
2.	$[\frac{1}{6},\frac{1}{2},\frac{1}{2};1,1,1\|1]$	$S_3(\Gamma_0(12),(\frac{-4}{\cdot}))$	12.3.d.a	$a(p)=S_6(p)\cdot F(\cdots)_p$	$p\equiv 1 \pmod 6$
3.	$[\frac{1}{8},\frac{1}{2},\frac{1}{2};1,1,1\|1]$	$S_3(\Gamma_0(64),(\frac{-8}{\cdot}))$	64.3.d.a	$a(p)=F(\cdots)_p$	$p\equiv 1 \pmod 8$
4.	$[\frac{1}{3},\frac{1}{3},\frac{2}{3};1,1,1\|1]$	$S_3(\Gamma_0(27),(\frac{-3}{\cdot}))$	27.3.b.b	$a(p)=F(\cdots)_p$	$p\equiv 1 \pmod 6$
5.	$[\frac{1}{4},\frac{1}{3},\frac{2}{3};1,1,1\|1]$	$S_3(\Gamma_0(36),(\frac{-4}{\cdot}))$	36.3.d.a	$a(p)=F(\cdots)_p$	$p\equiv 1 \pmod{12}$
6.	$[\frac{1}{6},\frac{1}{3},\frac{2}{3};1,1,1\|1]$	$S_3(\Gamma_0(108),(\frac{-3}{\cdot}))$	108.3.c.b	$a(p)=F(\cdots)_p$	$p\equiv 1 \pmod 6$
7.	$[\frac{1}{3},\frac{1}{4},\frac{3}{4};1,1,1\|1]$	$S_3(\Gamma_0(576),(\frac{-24}{\cdot}))$	576.3.h.b	$a(p)=S_{12}(p)\cdot F(\cdots)_p$	$p\equiv 1 \pmod{12}$
8.	$[\frac{1}{4},\frac{1}{4},\frac{3}{4};1,1,1\|1]$	$S_3(\Gamma_0(128),(\frac{-8}{\cdot}))$	128.3.d.c	$a(p)=S_4(p)\cdot F(\cdots)_p$	$p\equiv 1 \pmod 4$
9.	$[\frac{1}{6},\frac{1}{4},\frac{3}{4};1,1,1\|1]$	$S_3(\Gamma_0(576),(\frac{-24}{\cdot}))$	576.3.h.a	$a(p)=S_{12}(p)\cdot F(\cdots)_p$	$p\equiv 1 \pmod{12}$
10.	$[\frac{1}{3},\frac{1}{6},\frac{5}{6};1,1,1\|1]$	$S_3(\Gamma_0(432),(\frac{-4}{\cdot}))$	432.3.g.a	$a(p)=S_6(p)\cdot F(\cdots)_p$	$p\equiv 1 \pmod 6$
11.	$[\frac{1}{4},\frac{1}{6},\frac{5}{6};1,1,1\|1]$	$S_3(\Gamma_0(288),(\frac{-4}{\cdot}))$	288.3.g.a	$a(p)=F(\cdots)_p$	$p\equiv 1 \pmod{12}$
12.	$[\frac{1}{6},\frac{1}{6},\frac{5}{6};1,1,1\|1]$	$S_3(\Gamma_0(108),(\frac{-4}{\cdot}))$	108.3.d.a	$a(p)=S_6(p)\cdot F(\cdots)_p$	$p\equiv 1 \pmod 6$
13.	$[\frac{1}{5},\frac{1}{5},\frac{4}{5};1,1,1\|1]$	$S_3(\Gamma_0(25),\chi)$	25.3.c.a	$a(p)=F(\cdots)_p$	$p\equiv 1 \pmod 5$
14.	$[\frac{1}{2},\frac{1}{10},\frac{9}{10};1,1,1\|1]$	$S_3(\Gamma_0(20),(\frac{-20}{\cdot}))$	20.3.d.a	$a(p)=S_{10}(p)\cdot F(\cdots)_p$	$p\equiv 1 \pmod{10}$
15.	$[\frac{1}{2},\frac{1}{12},\frac{11}{12};1,1,1\|1]$	$S_3(\Gamma_0(24),(\frac{-24}{\cdot}))$	24.3.h.a	$a(p)=F(\cdots)_p$	$p\equiv 1 \pmod{12}$

EHMM Outline

Input: **Hypergeometric** HD

Output: Description of $L(HD;1;s)$ via an **Explicit Modular Form** $f^{\sharp}_{HD}$, if eligible.



EHMM by Michael Allen, Brian Grove, L., Fang-Ting Tu

- **Input**: $HD = \{\boldsymbol{\alpha}, \boldsymbol{\beta}\} = \{\boldsymbol{\alpha}^\flat \cup \{r\}, \boldsymbol{\beta}^\flat \cup \{s\}\}$
- **Conditions**:
  1. HD $\Rightarrow$ a differential
  $\Omega(t) = t^{r-1}(1-t)^{s-r-1}F(\alpha^\flat, \beta^\flat; t)dt = \sum a_n t^{n-1}dt, a_n \in \mathbb{Q}$
  and a specialization $t = t(q)$ yielding a "nice" q-expansion for

$$\Omega(t(q))q\frac{dt(q)}{dq} = c(q + \sum_{n\geq 2} b_n q^n)$$

  2. For each $p \equiv 1 \pmod{M}$, normalized "$H_p(\boldsymbol{\alpha}, \boldsymbol{\beta}; 1)$" $\in \mathbb{Z}$.
- **Conclusions**:
  1. An explicit description of b_n in terms of a modular form
  2. $L(HD; 1; s)$ has analytic/meromorphic continuation.
  3. Exact values of $L(HD; 1; s)$ at $s = 1$ or 2 in many cases

The **EHMM** method is

- applications oriented

  Conjectures of Dawsey-McCarthy

  Modularity of 4-dim'l repn's in EHMM2 (arXiv:2411.15116);
- applicable to many cases.

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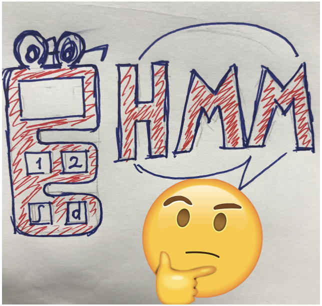
EHMM Calculator, an invitation



EHMM Calculator

- **Input**: $HD^\flat, r, s$ [e.g. $\{\{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\}, \{1, 1, 1\}\}, \frac{1}{2}, 1$]
  Either use given t in the database, or enter $t = t(q)$
- **Basic Outputs:**:
  q-expansion of $\Omega(t(q))\frac{dt(q)}{dq}$ [First few q-expansions]
  "normalized" $H_p(\boldsymbol{\alpha}, \boldsymbol{\beta}; 1; \omega)$
- **Verification**:
  - Whether $\Omega(t(q))\frac{dt(q)}{dq}$ behaves like a modular form
  - If yes, check its conjugates for compatibility
  - If yes, build a Hecke eigenform from conjugates
  - $\cdots$

EHMM Calc Demo

```
### Loding package
load('constructor.py');load('EHMMv3.sage')
### Construct a class
HD=EHMM([1/2,1/2,1/2,1/2],[1,1,1,1])
print ('Datum=',HD.alpha,HD.beta)
### Sample functions
print ("t's q-expansion")
print (HD.tq().O(6))
print (HD.q_expansion().O(10))
print ("First few coefficients of $f_{8.4.a.a}$")
print (Newforms(8,4)[0])
```

```
Datum= [1/2, 1/2, 1/2, 1/2] [1, 1, 1, 1]
t's q-expansion
-64*q - 1536*q^2 - 19200*q^3 - 167936*q^4 - 1160064*q^5 + O(q^6)
q - 4*q^3 - 2*q^5 + 24*q^7 - 11*q^9 + O(q^10)
First few coefficients of $f_{8.4.a.a}$
q - 4*q^3 - 2*q^5 + O(q^6)
```

In the remainder of this talk, we will demonstrate using

$$HD(r, s) = \left\{ \{\frac{1}{2}, \frac{1}{2}, r\}, \{1, 1, s\} \right\},$$

namely $HD^\flat = HD_2 = \{\{\frac{1}{2}, \frac{1}{2}\}, \{1, 1\}\}$ augmented by (r, s), based on classic results in analysis.

Schwarz map for $HD_m = \{\{\frac{1}{m}, \frac{m-1}{m}\}, \{1, 1\}\}$, $m = 2, 3, 4$

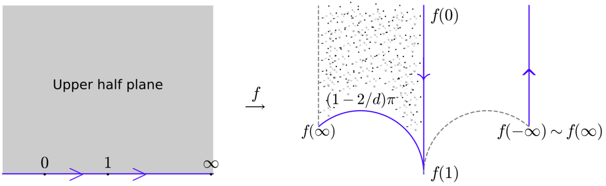
Both $F(HD_m; t)$ and $F(HD_m; 1-t)$ are annihilated by

$$\left[t(1-t)\frac{d^2}{dt^2} + (1-2t)\frac{d}{dt} - \frac{m-1}{m^2} \right] F(HD_m; t) = 0.$$

The **Schwarz map**

$$f(t) := \frac{\sqrt{-1}}{\kappa_m} \cdot \frac{F(HD_m; 1-t)}{F(HD_m; t)}$$

sends (where $\kappa_m = 1, \sqrt{3}, \sqrt{2}$ when $m = 2, 3, 4$ respectively)



Modular forms and hypergeometric functions

When $m = 2$, $f^{-1}(t)$ is the **modular lambda function**.

$$\lambda(\tau) = 16(q - 8q^2 + 44q^3 - 192q^4 + 718q^5 + \cdots),\ q = e^{\pi i \tau}.$$

Moreover,

$$\frac{d\lambda(\tau)}{\lambda(\tau)dq} = 1 - 8q + 24q^2 - 32q^3 + 24q^4 - 48q^5 + \cdots$$
$$= \sum_{n_i \in \mathbb{Z}} (-1)^{n_1+n_2+n_3+n_4} q^{(n_1^2+n_2^2+n_3^2+n_4^2)}$$

is a **weight-2 holomorphic** modular form.

From $HD(r,s)$ to modular forms $\mathbb{K}_2(r,s)$

By Euler's integral formula,

$$\frac{\Gamma(r)\Gamma(s-r)}{\Gamma(s)}\,{}_3F_2\left[\begin{matrix}1/2 & 1/2 & r \\ & 1 & s\end{matrix};\,1\right]$$

$$= \underbrace{\int_0^1 \lambda^{r}(1-\lambda)^{s-r-1}\,{}_2F_1\left[\begin{matrix}\frac{1}{2} & \frac{1}{2} \\ & 1\end{matrix};\,\lambda\right]\frac{d\lambda}{\lambda}}$$

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$$= \int_0^1 \underbrace{\lambda^r(1-\lambda)^{s-r-1}\,{}_2F_1\left[\begin{matrix}\frac{1}{2} & \frac{1}{2} \\ & 1\end{matrix};\ \lambda\right]\frac{d\lambda}{\lambda}}$$

$$\mathbb{K}_2(r,s) := 2^{1-4r}\,\overbrace{\lambda^r(1-\lambda^{s-r-1}\,{}_2F_1\left[\begin{matrix}\frac{1}{2} & \frac{1}{2} \\ & 1\end{matrix};\ \lambda\right]\frac{d\lambda(\tau)}{\lambda(\tau)dq}}$$

$$= \frac{\eta\left(\frac{\tau}{2}\right)^{16s-8r-12}\eta(2\tau)^{8s+8r-12}}{\eta(\tau)^{24s-30}},$$

where $\eta(\tau) := q^{1/24}\prod_{n\geq 1}(1-q^n)$ is the Dedekind eta function.

The $\mathbb{S}_2$ set

$$\mathbb{K}_2(r, s) = \frac{\eta\left(\frac{\tau}{2}\right)^{16s-8r-12}\eta(2\tau)^{8s+8r-12}}{\eta(\tau)^{24s-30}}.$$

For (r, s) in

$$\mathbb{S}_2 := \{(r, s) \mid 0 < r < s < 3/2,\ 24s, 8(r+s) \in \mathbb{Z}\},$$

$\mathbb{K}_2(r, s)$ is a holomorphic and congruence modular form.

Thus, we are endowed with **199 weight-3** congruence cusp forms, which are nice **building blocks**.

Conjugates

$$\frac{1}{M} \leftrightarrow \zeta_M = e^{2\pi i/M}$$

So primitive M^{th} roots of unity

$$\left\{\zeta_M^c \mid c \in (\mathbb{Z}/M\mathbb{Z})^\times\right\} \leftrightarrow \left\{\frac{c}{M} \in (\mathbb{Z}/M\mathbb{Z})^\times\right\}$$

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Two (r, s) and (r', s') in $\mathbb{S}_2$ are said to be **conjugates** if $\exists c \in (\mathbb{Z}/M\mathbb{Z})^\times$ such that $r' - cr, s' - cs \in \mathbb{Z}$.

We can group the elements in $\mathbb{S}_2$ into **conjugate clusters**.

Example

$\{(\frac{1}{8}, 1), (\frac{3}{8}, 1), (\frac{5}{8}, 1), (\frac{7}{8}, 1)\}$ form a conjugate cluster in $\mathbb{S}_2$.

Example $(\{(\frac{j}{8},1), j=1,3,5,7\})$

$$
\begin{aligned}
\mathbb{K}_2\left(\frac{1}{8},1\right)(16\tau) &= q - 3q^9 \underline{-6} q^{17} \cdots + 18q^{153} + \cdots \\
\mathbb{K}_2\left(\frac{3}{8},1\right)(16\tau) &= q^3 - q^{11} - 7q^{19} + 6q^{27} + \cdots \underline{-6} q^{51} + \cdots \\
\mathbb{K}_2\left(\frac{5}{8},1\right)(16\tau) &= q^5 + q^{13} - 4q^{21} - 3q^{29} + \cdots \underline{-6} q^{85} + \cdots \\
\mathbb{K}_2\left(\frac{7}{8},1\right)(16\tau) &= q^7 + 3q^{15} + 3q^{23} + 4q^{31} \cdots \underline{-6} q^{119} + \cdots
\end{aligned}
$$

The coefficients of $\mathbb{K}_2\left(\frac{1}{8},1\right)(16\tau)$ are multiplicative.

They are all in the Hecke orbit of 256.3.*c*.*g* (a twist of 64.3.*d*.*a*). We call such a conjugate cluster to be **Galois**.

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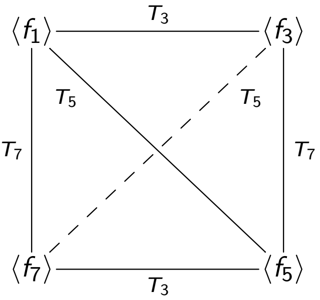
Hecke actions on Galois clusters, by example

Let $f_i := \mathbb{K}_2(\frac{i}{8}, 1)(16\tau)$, T_p denotes the p^{th} Hecke operator. Then



Moreover,

$$T_p f_1 = h_p \cdot f_p, \quad \text{for } p = 3, 5, 7$$

with $h_p = -12, 48, -64$, for $p = 3, 5, 7$ respectively.

EHMM for $HD(r,s)=\{\{\frac{1}{2},\frac{1}{2},r\},\{1,1,s\}\}$

Theorem (Allen, Grove, L. and Tu)

1) If $(r,s)\in\mathbb{S}_2$ is in a **Galois** *cluster, then* **the two conditions are satisfied**. *For each $p\equiv 1 \pmod{M}$, normalized*

$$\text{“}H_p\left(\{\frac{1}{2},\frac{1}{2},r\},\{1,1,s\};1\right)\text{”}=a_p(f^{\sharp}_{HD(r,s)}),$$

where $f^{\sharp}_{HD(r,s)}$ is a weight-3 Hecke eigenform.

2) $f^{\sharp}_{HD(r,s)}$ is an **explicit** *linear combination of functions $\mathbb{K}_2(r,s)$ in the conjugate cluster; $L(f^{\sharp}_{HD(r,s)},1)$ is a linear combination of $P(r,s)$, where* $P(r,s):=\frac{\Gamma(r)\Gamma(s-r)}{\Gamma(s)}\,{}_3F_2\left[\begin{matrix}1/2 & 1/2 & r\\ & 1 & s\end{matrix};\,1\right]$.

Example $(\{\{\frac{1}{2}, \frac{1}{2}, \frac{j}{8}\}, \{1, 1, 1\}\})$

1.

$$\begin{aligned} f^{\sharp}_{\{\{\frac{1}{2},\frac{1}{2},\frac{j}{8}\},\{1,1,1\}\}} &= f_{256.3.c.g} \\ &= f_1 + \sqrt{h_3} \cdot f_3 + \sqrt{h_5} \cdot f_5 + \frac{\sqrt{h_3 h_5}}{3} \cdot f_7. \end{aligned}$$

2.

$$\begin{aligned} L(f_{256.3.c.g}, 1) = \sqrt{2}(P(1/8, 1) - \sqrt{-1}P(7/8, 1)) \\ + \sqrt{-6}(P(3/8, 1) + \sqrt{-1}P(5/8, 1)). \end{aligned}$$

Esme Rosen

- *L-values of certain weight 3 Modular Forms and Transformations of Hypergeometric Series* [arXiv:2412.07054] [Using a Coxeter group acting on $\mathbb{S}_2$, all Galois and non-Galois cases of the $\mathbb{K}_2$-families are classified.]
- *'Mixed' CM structures associated to certain Hypergeometric Motives* (in preparation) [Non-Galois cases of $\mathbb{K}_2$ possess 'Mixed' CM structures.]
- *Modular Forms and Certain ${}_2F_1(1)$ Hypergeometric Series* [arXiv:2502.08760] [A complete classification of the weight-2 $\mathbb{K}_1$ families.]

Brian Grove

- *On Some Hypergeometric Modularity Conjectures of Dawsey and McCarthy* [arXiv:2507.19971]
[A complete classification of the weight-3 $\mathbb{K}_3$ and twisted $\mathbb{K}_3^{\mathsf{kmr}}$ families.]

Thank you!

https://www.math.lsu.edu/∼llong/EHMMCalc.html